# Complex Numbers

*Third Level\ Phys. Dep.*

Lecture One

The aim of the lecture is to familiarize the student with the following:

1- Concept of Complex Numbers

2- Geometric Represention of Complex Numbers

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# COMPLEX ANALYSIS FORMULA SHEET

## Real number representation

Real number represent graphically in one dimension ( either horizontal or vertical) as shown.

Due to *Quadratic* Algebric Equation ;

$$ax^2 + bx + C = 0$$

The solution with be $x_1$ and $x_2$ . The square root of ( $\sqrt{b^2 - 4ac}$ ) may be ( posative , negative or zero )

**The *negative* value will be expressed as ( *complex number*)**

## Complex Numbers represent by:

1. ***Rectangular*** **coordinate representation**

$$z = x + iy \implies \quad as\ a\ point\ with\ (x\ ,y)$$

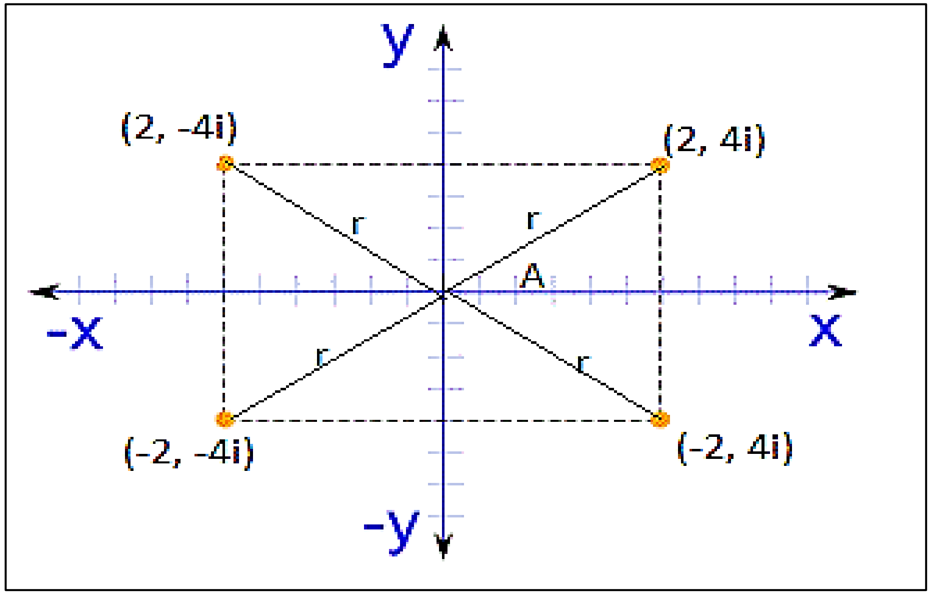
$\boldsymbol{x}$ *and* $\boldsymbol{y}$ are real numbers .

$\boldsymbol{x}$ are the real part of $z \implies x = Re(z)$

$\boldsymbol{y}$ are the imaginary part of $z \implies y = Im(z)$

$$z = Re(z) + Im(z).\boldsymbol{i}$$

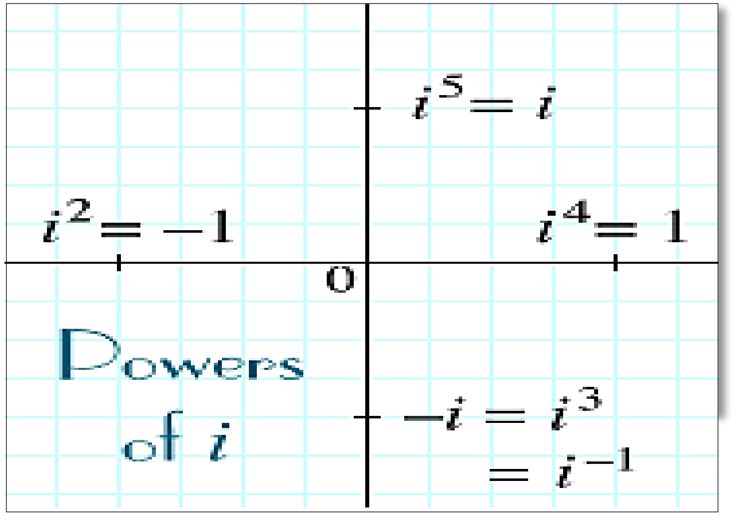
***Argand Plane* or *Complex Plane***



## Complex unit

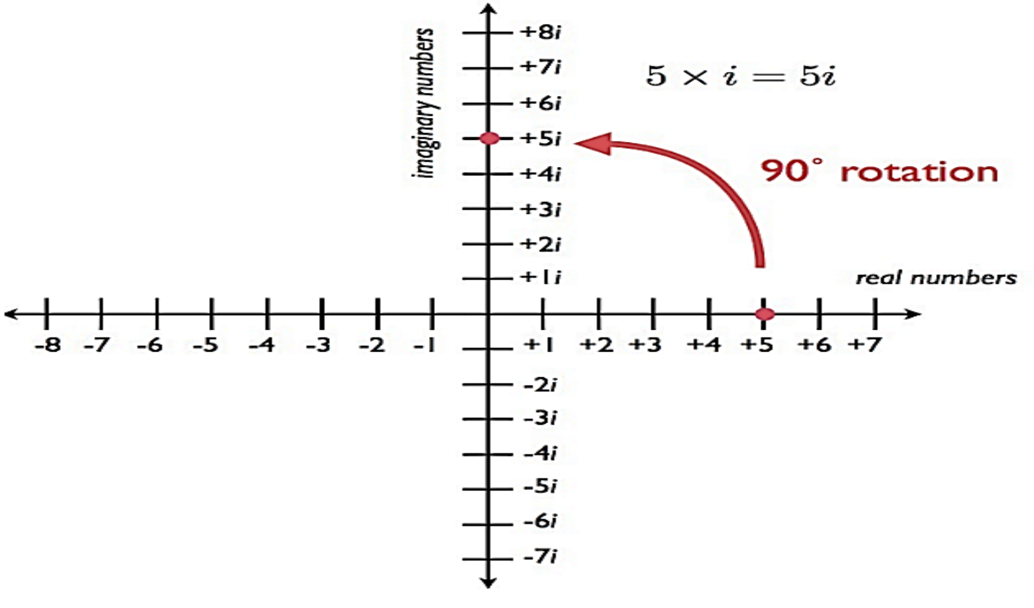
Now,

$$i = \sqrt{-1}$$
$$i^2 = -1$$




$$i^5 = i^2.i^2.i = (i^2)^2.i = (-1)^2.i = +i$$

$$i^{101} = (i^2)^{50}.i = (-1)^{50}.i = +i$$



## 2. ***Polar*** coordinate representation

$z = re^{i\theta}$ specified by $(\boldsymbol{r}, \boldsymbol{\theta})$

$z = r[\cos\theta + i\sin\theta]$ specified by $(r, \theta)$ ***known as Euler's formula***

$z = r\angle\theta$

$$z = \boldsymbol{r}\angle\boldsymbol{\theta}$$

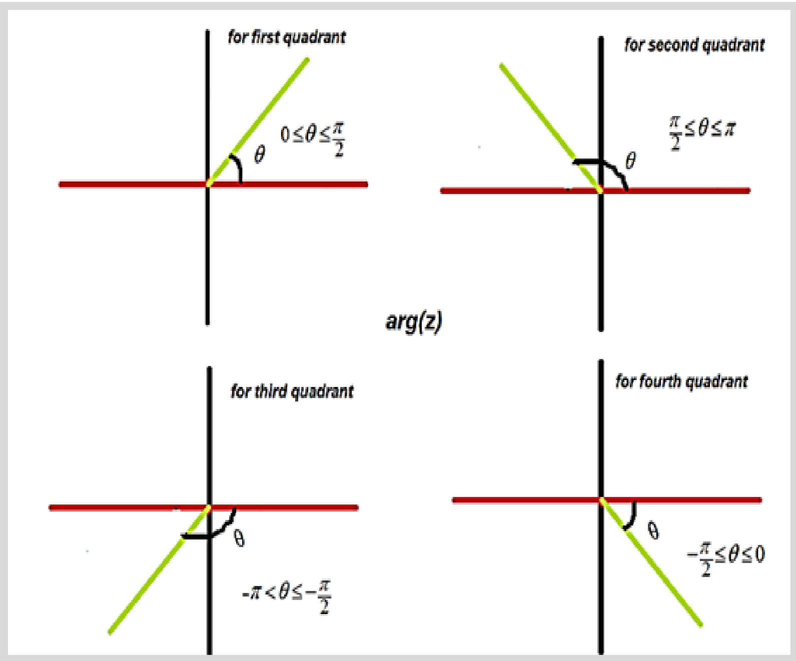
- **Length** of $z$
- **Amplitude** $z$
- **Modulus** $z$
- **Absolute value**

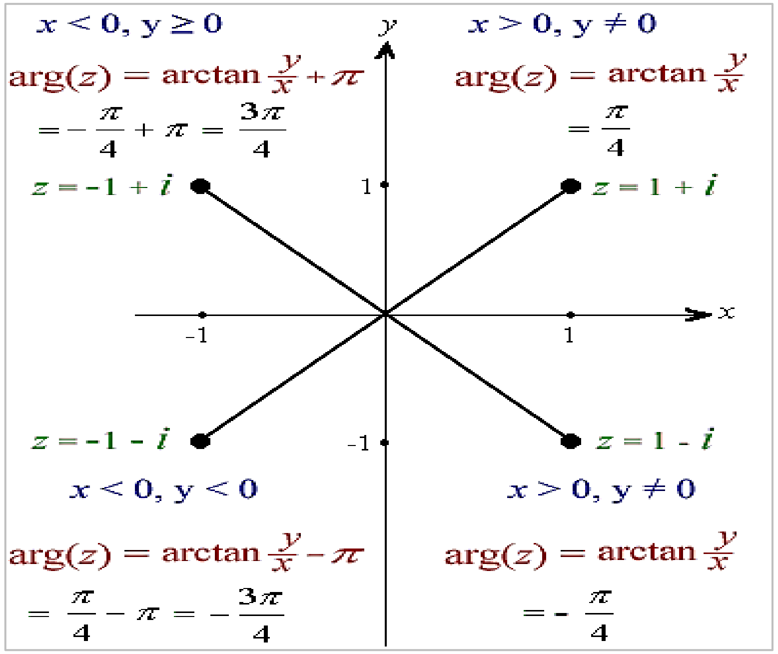
$$|z| = \sqrt{x^2 + y^2} = r$$

$r$ is the " ***radius of circle*** " **centered** at orign.

$\theta$ is called '***angle*** " or , " ***argument***' or '***phase*** " represent the direction of $Z$ and can be evaluated by :

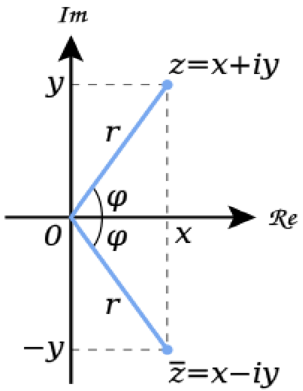
$$\theta = tan^{-1}(\frac{y}{x})$$

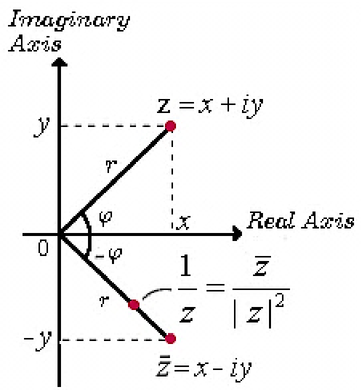


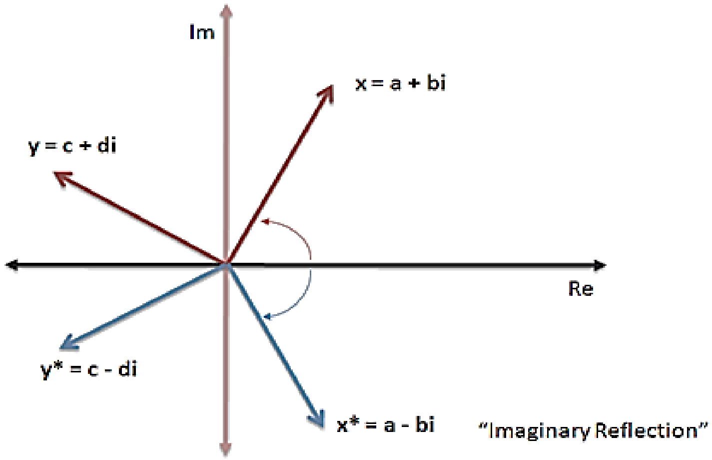




**Complex conjugate of** $z$ Represented by $\bar{z}$ or $z^*$

$$\bar{z} = x - iy \implies as\ a\ point\ with\ (x\ , -y)$$




## Inflection of $z$

$$-z = -x - iy \implies \quad as\ a\ point\ with\ (-x\ ,-y)$$

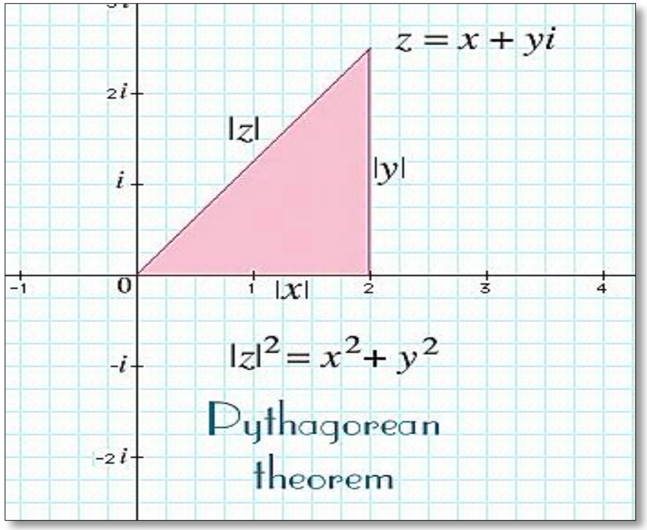
## Absolute value of $z$

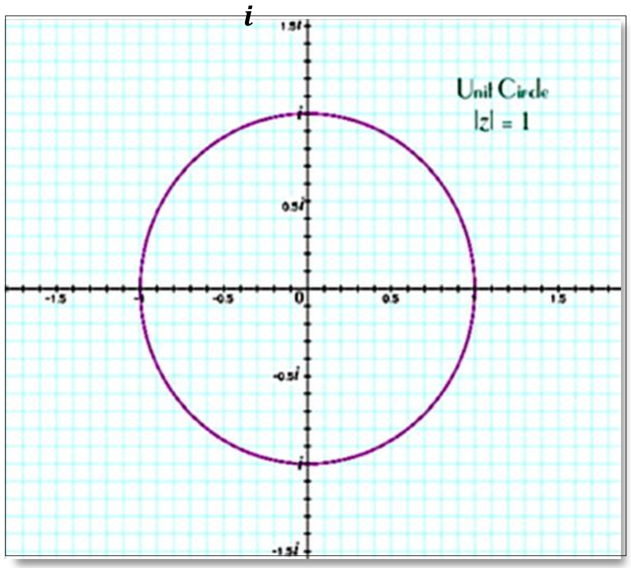
*Also called* ,
- **Length** of $z$
- **Amplitude** $z$
- **Modulus** $z$

$$|z| = \sqrt{x^2 + y^2} = r$$

$r$ is the " ***radius of circle*** "

**centered** at $(x\ ,y)$.




## Distance between $z_1$ *and* $z_2$

Represented by $|z_1 - z_2|$ and given by :

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

## Power of $z$

*Or* ,

$$z^n = r^n e^{(in\theta)}$$

$$(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$$

"*De Moivers theorem*"

$$z = x + iy = |z|(cos\,\theta + isin\theta) = |z|e^{i\theta}$$

## ***PASCAL TRIANGLE***

$$(a+b)^0 = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b \; \ldots\ldots\text{.etc}$$

### Roots of $z$

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \qquad k=0,1,2,3,\ldots,(n-1)$$

$z_0, z_1, z_2 \ldots$ *the roots of* $z$

# Complex Numbers

## المحاضرة الثانيــــة

### ***Examples***

1. Verify

a) $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=-2i$

b) $\left(\frac{1+2i}{3-4i}\right)+\left(\frac{2-i}{i}\right)=\frac{-2}{5}$

$$\Longrightarrow \frac{1+2i}{3-4i}*\frac{3+4i}{3+4i}+\frac{2-i}{5i}*\frac{-5i}{-5i}$$

$$=\frac{3+4i+6i-8}{25}=\frac{-10}{25}=\frac{-2}{5}$$

c) $\frac{5}{(1-i)(2-i)(3-i)}=\frac{1}{2}\,i$

$$\frac{5}{(1-i)(2-i)(3-i)}=\frac{5}{(1-3i)(3-i)}=\frac{5}{3-i-9i-3}=\frac{5}{-10i}=\frac{-i}{2i^2}$$

$$=+\frac{i}{2}\quad \text{since } i^2=-1$$

2. Simplify $\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)$

*Solution:*

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)=\frac{1}{2+3-3i+2i}=\frac{1}{5-i}*\frac{5+i}{5+i}=\frac{5+i}{25-1}=\frac{5+i}{24}$$

$$=\frac{5}{24}+\frac{1}{24}i$$

**Question :** Express $X, Y, and\ |z|^2$ in terms of $Re(z)$ and $Im(z)$

*Solution:*

$$z=x+iy \quad \ldots(1)$$

$$\bar{z}=x-iy \quad \ldots(2)$$

Add (1) and (2) ,

$$z + \bar{z} = 2x \Longrightarrow \boxed{X \equiv Re(z) = \frac{z + \bar{z}}{2}}$$

Now, subtract (1) and (2),

$$z = x + iy$$
$$\underline{\bar{z} = x - iy}$$

$$z - \bar{z} = 2yi \Longrightarrow \boxed{Y \equiv Im(z) = \frac{z - \bar{z}}{2i}}$$

$$|z|^2 = zz^* = \boxed{x^2 + y^2} = \big(Re(z)\big)^2 + \big(Im(z)\big)^2$$

**Example:** Simplify

$$\frac{-1+3i}{\quad 2-i} * \frac{2+i}{\quad 5} = -1 + i$$

**Basic of algebraic properties of *z*, verify a few algebraic properties of *z*.**

1. The commutative laws.

$$z_1 \pm z_2 = z_2 \pm z_1, \qquad z_1 z_2 = z_2 z_1$$

2. The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Now,

$$\frac{1}{z_1 z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right), \quad (z_1 \neq 0, z_2 \neq 0, z_1 z_2 \neq 0)$$

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \;, \quad \frac{z_1}{z\ z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right), \quad (z_3 \neq 0, z_4 \neq 0, z_3, z_4 \neq 0)$$

## Absolute value of $Z$

With that interpretation in mind, we introduce the ***length*** , ***amplitude*** , ***absolute value*** or ***modulus*** of the complex number its the length when thinking of it as a vector :

If $z = x + iy$ then $|z| = \sqrt{X^2 + y^2}$

$|z|$ is the distance between the point $(x, y)$ and the origin.

**Question** : what are the main difference between the absolute value of real numbers and complex numbers.

$$|z| = \sqrt{x^2 + y^2}$$

$|z_1 - z_2| =$ is the distance between the points representing the complex numbers $z_1$ a . When $y = 0$, $|z| = x$ usual absolute value in the real numbers system. from the point$(x_2, y_2)$ to the point $(x_1, y_1)$.

***Example***:

Compute the absolute value for each of the complex numbers : $z_1 = 1 + 2i$ , $z_2 = -$

.

*Solution*:

$$z_1 = x_1 + iy_1 \Longrightarrow z_1 = 1 + 2i \Longrightarrow |z_1| = \sqrt{1+4} = \sqrt{5} z_2$$

$$|z_2| = \sqrt{9+1} = \sqrt{10}$$

$$|z_1| = \sqrt{5} \cong 2, \qquad |z_2| = \sqrt{10} \cong 3$$

The distance between the points representing the complex numbers $z_1$and $z_2$

$$\boxed{|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

***Example***: If $z_1 = 1 + i, z_2 = 2 - 3i$, find $|z_1 - z_2|$

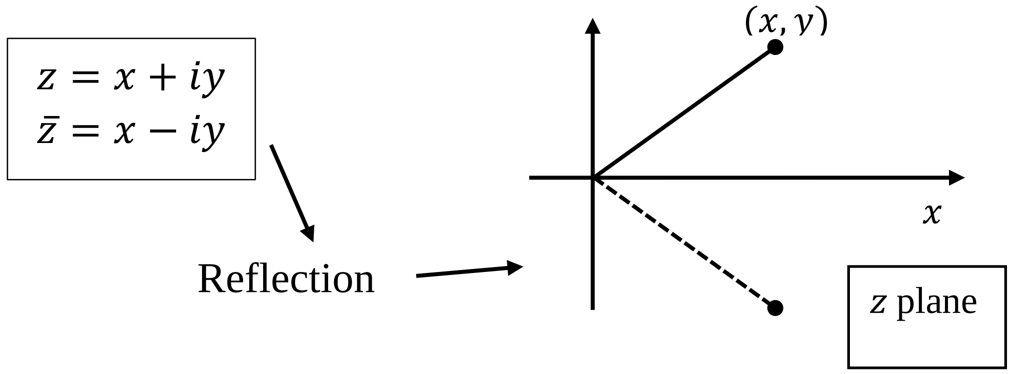
*Solution*:

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(1-2)^2 + (1+3)^2} = \sqrt{1^2 + 4^2} = \sqrt{17}$$

## Geometric Representation of *z*

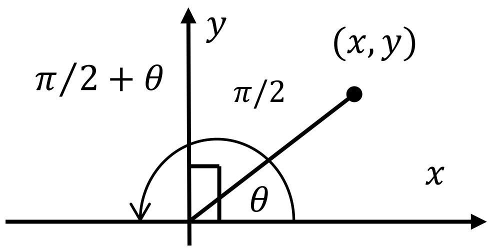
1- Reflection on the real axis.




2- Rotation of *z*

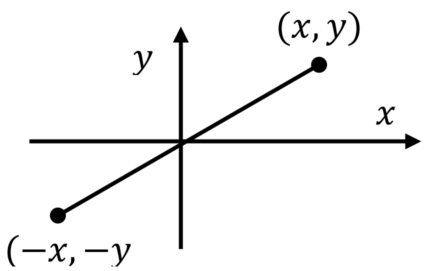
If $z$ is non-zero then $iz$

$$z = x + iy$$
$$iz = ix - y$$
$$e^{\pi/2\, i} e^{i\theta} = e^{i(\theta + \pi/2)}$$




3- Inflection of *z*

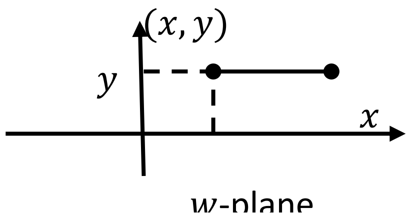
$$z = x + iy$$
$$-z = -x - iy$$



## *Operations*

4- Translation of $z$

$$z = x + iy$$
$$w = 1 + z = 1 + x + iy$$




*Operations in a comlpex plane*

# Complex Numbers

2022

**المحاضرة الثالثة :**

(1) رسم الاعداد العقدية ، جمعها طرحها بالرسم

(2) تطبيقات القيمة المطلقة للاعداد العقدية

## *Cartesian Coordinates*

$$z = x + iy$$

It is natural to associate the complex number $z = x + iy$ with a point in the plane whose carteian coordinates are $x$ and $y$.

**a)** The number $2 + i$ isrepresented by the point (2,1).

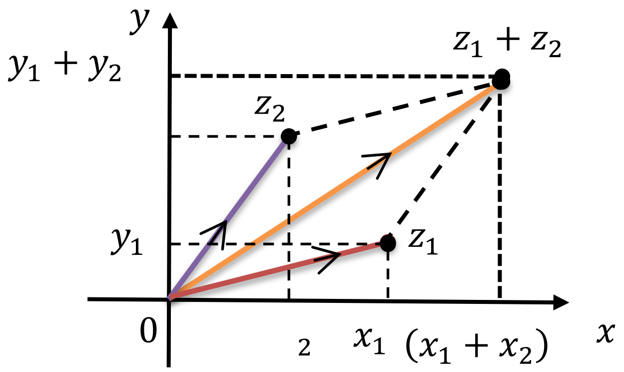
**b)** The number $z$ can also be thought of as the directed line segment, or vector, from the origin to the point $(x, y)$.

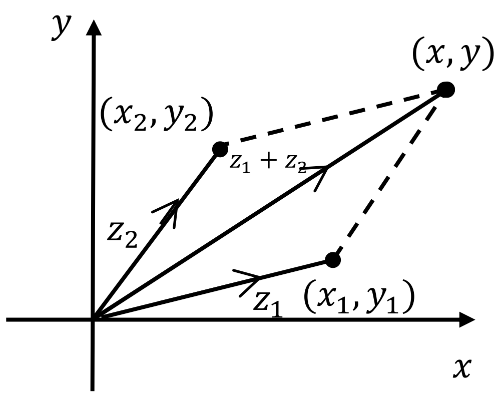
***Question*:** How can one represent the complex number?

Complex number may be represented either by a point in the plane, or as a vector starting from 0 to $(x, y)$.

a) It also corresponds to a vector with those coordinates as its components.

***Question*** **:** Represent vectorially $z_1 + z_2$



**NOTE**: The addition of $(z_1 + z_1)$ is geometrically represented by the parallelogram law.

## Exercises

**1.** Locate the numbers $z_1 + z_1$ and $z_1 - z_2$ vectorially, when:

**a)** $z_1 = 2i, \ \ z_2 = \frac{2}{3} - i$

**b)** $z_1 = (-\sqrt{3}, 1), \ z_2 = (\sqrt{3}, 0)$

**c)** $z_1 = (-3,1), \ z_2 = (1,4)$

**d)** $z_1 = x_1 + iy_1, \ z_2 = x_1 - iy_1$

## Solution

**a)** $z_1 = 2i$ $\qquad$ $z_1 = 2i$

$z_1 + z_2 = \frac{2}{3} + i$ $\qquad$ $z_1 - z_2 = -\frac{2}{3} + 3i$

**b)** $z_1 = (-\sqrt{3}, 1), \ z_2 = (\sqrt{3}, 0)$

$z_1 = -\sqrt{3} + i$ $\qquad$ $z_1 = -\sqrt{3} \pm i$

$z_2 = \sqrt{3}$ $\qquad$ $\mp z_2 = \mp\sqrt{3} + 0i$

$z_1 + z_2 =$ $\qquad$ $z_1 - z_2 = -2\sqrt{3} + i$

**c)** $z_1 = (-3,1), \ z_2 = (1,4)$

$z_2 = 1 + 4i$ $\qquad$ $\mp z_2 = \mp 1 \mp 4i$

$z_1 + z_2 = -2 + 5i$ $\qquad$ $z_1 - z_2 = -4 - 3i$

**d)**

$$z_1 = x_1 + iy_1 \qquad\qquad z_1 = x_1 + iy_1$$

$$\underline{z_2 = x_1 - iy_1 \qquad\qquad \mp z_2 = \mp x_1 \pm iy_1}$$

$$z_1 + z_2 = 2x_1 \qquad\qquad z_1 - z_2 = 2iy_1$$

***Question***: Show that the complex numbers corresponding to the points on the circle with center (0,1) and radius 3, satisfy the equation $|z - \blacksquare| = 3$ and conversely. **Or/** ***we refer to the set of points simply as the circle*** $|z - i| = 3$

$$z = x + iy$$

$$|z - i| = |x + iy - i| = |x + i(y-1)| = 3$$

$$= \sqrt{x^2 + (y-1)^2} = 3 \Longrightarrow \text{Circle of radius 3 centered at } (\mathbf{0}, \mathbf{1})$$

***Question*** : Plot $|z - i| = 3$, if $z = 2 + 3i$ ,

$$|2 + 3i - i| = 3$$

$$|2 + 2i| = 3 \Longrightarrow \sqrt{2^2 + 2^2} = 4$$

$$x = 2, y = 2, r = 3$$

## *Absolute Value*

- $|z_1 z_2| = |z_1||z_2|$ ,
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z}$

## *Conjugate of complex numbers* :

- sum of the conjugate $\overline{z_1 + z_2} = \overline{z} + \overline{z}$
- Subtraction of the conjugate $\overline{z_1 - z_2} = \overline{z} - \overline{z}$
- Multiplication of the conjugate $\overline{z_1 z_2} = \overline{z}\,\overline{z}$
- Division of the conjugate $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\ }$

## 2. Show that

**a)** $\overline{\overline{z} + 3i} = z - 3i$

**b)** $\overline{iz} = -i\,\overline{z}$

**c)** $\overline{\frac{(2+i)^2}{3-4i}} = 1$

**d)** $\left|\frac{2\overline{z}+5}{\sqrt{2}-i}\right| = \sqrt{3}|2z+5|$

***Solution***:

**a)** $\overline{\overline{z} + 3i} = z - 3i$, or since $\overline{\overline{z}} = z$

$$\overline{\overline{(x+iy)} + 3i} = \overline{(x-iy) + 3i} = \overline{(x-iy)} + \overline{3i} = (x+iy) - 3i = z - 3i$$

**b)** $\overline{iz} = \overline{i}\,\overline{z} = -i\,\overline{z}$

**c)** $\overline{\frac{(2+i)^2}{3-4i}} = 1$

$$\frac{\overline{(2+i)^2}}{\overline{3-4i}} = \frac{\overline{3+4i}}{3-4i} = \frac{3-4i}{3-4i} = 1$$

**d)** $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}|2z+5|$, since $|z_1 z_2| = |z_1||z_2|$

***L.H.S***

$$\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = |(2\overline{z}+5)||(\sqrt{2}-i)|$$

$$|\sqrt{2}-i| = \sqrt{\left(\sqrt{2}\right)^2 + (-1)^2} = \sqrt{2+1} = \sqrt{3}$$

$$|(2\overline{z}+5)| = \left|\left(2\overline{(x+iy)}+5\right)\right| = |(2(x-iy)+5)| = |(2x-2iy+5)| =$$

$$\left|((2x+5)-2iy)\right| = \sqrt{(2x+5)^2 + (2y)^2}$$

$$= 4x^2 + 20x + 25 + 4y^2$$

$|z| = \sqrt{x^2+y^2}$

***R.H.S***

$$\sqrt{3}|2z+5| = \sqrt{3}|(2x+2iy+5)| = \sqrt{3}|(2x+5)+2iy|$$

$$= \sqrt{3}\sqrt{(2x+5)^2 + (2y)^2}$$

$$L.H.S \equiv R.H.S$$

# Complex Numbers

2022

المحاضرة الرابعة :

1. الاعداد العقدية بالنظام القطبي .. أمثلة تطبيقية

## ***Polar* Form or *Trigonometric* Form of a Complex Number**

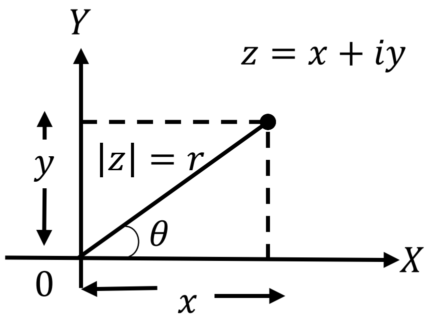
A complex number $z$, when written as

$$z = x + iy \quad or, \qquad z = x + jy$$

is said to be expressed in *rectangular form* ,also known as Cartesian coordinates .

But is also may be expressed in *polar form*

$$z = r\angle\theta$$




*Geometrically*, $|z|$ is the distance of the point from the origin $r$ or can be represent by $(A)$ .The number $r$ ( or $A$ ) is the length (or amplitude, modulus of the vector representing $z$; that is,

$$\boxed{r = |z|} = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \quad (r \geq 0) \ , \ \theta = tan^{-1}(\frac{y}{x})$$

## Conjugate of a Complex Number

The conjugate $z^*$ or $\bar{z}$ of a complex number $z$ is obtained by ***changing the sign of the imaginary part of the number.***

Conjugate of $z = z^* = Re\{z\} - iIm\{z\} = r\angle - \theta$

Now mathematical operation of complex number :

1. Addition $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
3. Multiplication $z_1 z_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$
4. Division $\frac{z_1}{z}$ $-$

5. Reciprocal $\frac{1}{z} = \frac{1}{r}\angle -\theta$
6. Square root $\sqrt{z} = \sqrt{r}\angle\ {}^{\theta}/_{2}$
7. Complex conjugate $z^* = x - jy = r\angle -\theta$

$$+90^o = +\sqrt{-1} = +j = 1\angle +90^o = 0 + j1$$

$$-90^o = -\sqrt{-1} = -j = 1\angle -90^o = 0 - j1$$

$$\mp 180^o = (\ \sqrt{-1}\ )^2 = -1 = 1\angle \mp 180^o = -1 + j\,0$$

## Converting Polar Form into Rectangular Form, ( P→R )

(From **P**olar form to **R**ectangular form)

$$6\angle 30^o = x + jy$$

However,

$$x = A\cos\theta \quad , y = A\sin\theta$$

Therefore,

$$6\angle 30^o = (6\cos\theta) + j(6\sin\theta)$$

$$=(6\cos\theta) + j(6\sin\theta)$$

$$=(6\cos 30^o) + j(6\sin 30^o)$$

$$=(6 \times 0.866) + j(6 \times 0.5)$$

## Converting Rectangular Form into Polar Form ( R→P )

(From **R**ectangular form to **P**olar form)

$$(\mathbf{5.2 + j3}) = \boldsymbol{A\angle\theta}$$

Where $\boldsymbol{A} = \sqrt{\mathbf{5.2^2 + 3^2}} = \mathbf{6}$ and $\boldsymbol{\theta} = \boldsymbol{tan^{-1}}\frac{\mathbf{3}}{\mathbf{5.2}} = \mathbf{30^o}$

Hence , $(\mathbf{5.2 + j3}) = \mathbf{6\angle 30^o}$

# Complex Numbers

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## Multiplication in Polar Form

$$Z_1 Z_2 = (A_1 A_2)\angle(\theta_1 + \theta_2)$$

Multiplying together 6 $\angle 30^o$ and 8 $\angle -45^o$ in polar form gives us:

$$Z_1 Z_2 = (6 \times 8)\angle(30^o + (-45^o)) = 48\angle -15^o$$

## Division in Polar Form

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right)\angle\theta_1 - \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{6}{8}\right)\angle 30^o - (-45^o) = 0.75\angle 75^o$$

## Polar Coordinates

Let $r$ and $\theta$ be polar coordinates of the point $(x, y)$ corresponding to a nonzero complex number $z = x + iy$

Since, $x = r\cos\theta$ , $y = r\sin\theta$

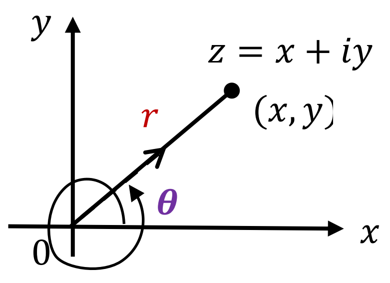
$$\boxed{z = r(\cos\theta + i\sin\theta)} \quad , \qquad \textit{polar form of number } z$$

For example,

$$1 + i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$$

The number $\theta$ is called an *argument* of $z$, (also called, '*phase*' and '*angle*')

$\theta$ is the directed angle measured from the positive $x$-axis to a point $p$ and called (*argument* of z) and is denoted by : arg ; $\theta = \arg z_1$




Hence the angles will be measured in *radian* and positioned in the *counterclockwise sense*. $\boxed{\tan\theta = \frac{y}{x}}$

# Complex Numbers

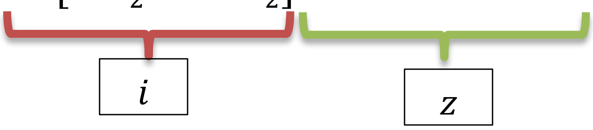
2022

### *Important identity*

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

***Question*** : Find the value of ( $iz$ )

$$iz = (1)\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] r(\cos\theta + i\sin\theta)$$




$$iz = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

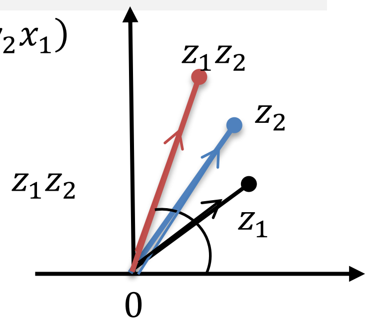
$iz$ has an important applications in Optics i.e. (*circular polarization*).

***Example:***

$\theta_1 = \arg(1+i) \quad , \theta_1 = arg(-1-i) \Longrightarrow \tan\theta_1 = \tan\theta_2 = 1$....**PLOT!!**

***Question*** : Use Cartesian coordinate system representation of z , simplified the multiplication of two different complex numbers.

$z_1 z_2 = (x_1 + iy_2)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + y_2 x_1)$

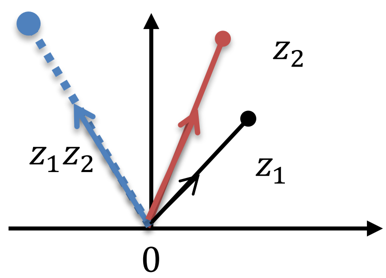



$\mathbf{(1+i)(2+3i) = 2 + 5i - 3 = -1 + 5i}$

$z_1 = (1+i) \rightarrow x_1 = 1, y_1 = 1, \ r_1 = \sqrt{2}, \ \ \theta_1 = 45°$ ,

$\tan^{-1}\frac{3}{2}$ or $\theta_2 = 56.31°$,

For the ***final results*** $-1 + 5i$, will be :




$r = 5.1 \quad , \theta = \tan^{-1} -\frac{5}{1} \ , \theta = 101.31°$

# Complex Numbers

2022

المحاضرة الخامسة :

1. الدوال الاسية العقدية وخواصها مع امثلة .
2. قوى الاعداد العقدية مع امثلة .
3. جذور العدد العقدي مع أمثلة .

## *Exponential Function* الدالة الاسية العقدية

The polar form is $z = r(\cos\theta + i\sin\theta)$ ,

or , $z = re^{i\theta}$

Where $\boxed{e^{i\theta} = (\cos\theta + i\sin\theta)}$, is ***Euler's formula*** , and $e^{i\theta}$(*complex exponential function*)

***NOTE:***

$$(1)\ z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$(2)\ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

***Question*** **: Relate trigonometric function with complex exponential function**

الربط بين الدوال الاسية العقدية والدوال المثلثية الحقيقة

**Proof:** $\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \ldots(1)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad \ldots(2)$$

Adding (1) and (2) : $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Longrightarrow \boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$

subtracting (1) and (2)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

# Complex Numbers

2022

***Powers*** of complex numbers

$\boxed{z^n = r^n e^{in\theta}}$, $(n = 0, \pm 1, \pm 2, \dots)$ , **where** $\left(e^{i\theta}\right)^n = e^{in\theta}$

Or $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = \cos\phi + i\sin\phi}$,

$(n = 0, \pm 1, \pm 2, \dots)$ $\quad$ (***de Moivre's theorem!!***)

Where $\boxed{\phi = n\theta}$

## Roots of complex number z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \qquad \boxed{k = 0,1,2,\dots,n-1}$$

***Example***

1. Solve the equation $z^3 + 1 = 0$, or in other words
2. Find $z$ for the equation $z = (-1)^{\frac{1}{3}}$
3. Find the third roots(***ONLY***) of the equation $z^3 + 1 = 0$

***Solution***

$$z^3 = -1, = (-1)^{\frac{1}{3}},$$

$$x = -1, y = 0, r = \sqrt{x^2 + y^2} \Longrightarrow r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Longrightarrow \quad \theta = \pi, \qquad n = 3, \qquad k = 0,1,2$$

$1^{st}\ root\ \boxed{k = 0}$, $z_1 = (1)\left[cos\frac{\pi+2(0)\pi}{3} + i\,sin\frac{\pi+2(0)\pi}{3}\right]$

$$= \frac{1}{2} + i0.866 = 0.5 + i0.866$$

$2^{nd}\ root\ \boxed{k = 1}$,

$$z_2 = \left[cos\frac{\pi + 2\pi}{3} + i\,sin\frac{\pi + 2\pi}{3}\right]$$

$$= \cos\pi + i\sin\pi = -1 + 0i$$

$3^{rd}\ root\ \boxed{k = 2}$,

$$z_3 = \left[cos\frac{\pi + 4\pi}{3} + i\,sin\frac{\pi + 4\pi}{3}\right]$$

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$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = 0.5 - 0.866i$

The 3rd root $k = 2$ was $(0.5\ ,-0.866)$

## *Examples*

**1.** Find one value of $\arg(z)$ when

**a)** $z = \frac{-2}{1+i\sqrt{3}}$ , **b)** $z = \frac{i}{-2-2i}$ , **c)** $z = \left(\sqrt{3}-i\right)^6$

***Solution***:

**a)** $z = \frac{-2}{1+i\sqrt{3}}$ , $\frac{-2}{1+i\sqrt{3}}\ \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{(-2)(1-\sqrt{3}i)}{1+3}$

$= \frac{-2+2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

$\therefore \boldsymbol{x} = -\frac{\mathbf{1}}{\mathbf{2}} = -\mathbf{0.5}$

$y = \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{3}} = \mathbf{0.866} \Longrightarrow \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1/_2\sqrt{3}}{/}\right)$

$\theta = \tan^{-1}(-\sqrt{3}) \Longrightarrow \boxed{\theta = -60}, \theta = 180° - 60° = 120°$

**b)** $\frac{i}{-z-2i}$ ,

$\frac{i}{-z-2i} = \frac{i}{-2-2i} * \frac{-2+2i}{-2+2i} = \frac{i(-2+2i)}{\quad}$ 8

$\frac{i}{-z-2i} = \frac{-2i+2}{8}$

$\frac{i}{-z-2i} = \frac{1-i}{4} = \frac{\mathbf{1}}{\mathbf{4}} - \frac{\mathbf{1}}{\mathbf{4}}i$

$x = \frac{1}{4}, y = -\frac{1}{4} \Longrightarrow \theta = \tan^{-1}\left(\frac{-1/4}{/}\right) = \tan^{-1}(-1) = -45° = 315° = -\frac{\pi}{4} \equiv \frac{3\pi}{4}$

**c)** $z = \left(\sqrt{3}-i\right)^6$

$x = \sqrt{3}$

$y = -1,\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$

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$$n\theta = 6 * \theta = -30° * 6 = -180° = 360° - 180° = \pi$$

**2.** Use the polar form to show that

a) $i(1 - i\sqrt{3})(\sqrt{3} + i) = 2 + i2\sqrt{3};$

b) $5i(2 + i) = 1 + 2i;$

c) $(-1 + i)^7 = -8(1 + i);$

d) $(1 + i\sqrt{3})^{-10} = 2^{-11}(-1 + i\sqrt{3}).$

## *Solution*

a) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$x = \sqrt{3}$

$y = +1$ ,  $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$L.H.S$  $(\sqrt{3} + i)(\sqrt{3} + i) = 3 - 1 + 2\sqrt{3}i = 2 + 2\sqrt{3}i$

b) $5i/(2 + i)$

$$\frac{5i}{2+i} = \frac{5i}{2+i} * \frac{2-i}{4+1} = (\frac{5(1+2i)}{5} = 1 + 2i$$

c) $(-1 + i)^7 = -8(1 + i)$

L.H.S  $(-1 + i)^7$

$x = -1$

$y = 1,$  $\theta = \tan^{-1}\left(\frac{y}{x}\right) \Longrightarrow \theta = \tan^{-1}(-1) \Longrightarrow, \theta = -45°,$

Then $\theta = 180° - 45° = 135°$ , Or, $\theta = \frac{3\pi}{4}$

$$7\theta = 7 * \frac{3\pi}{4} = \frac{21}{4}\pi$$

$$e^{i.\frac{21}{4}\pi} \overset{?}{=} e^{\cdot\frac{\pi}{4}i}$$

R.H.S  $-8 - 8i \Longrightarrow x = -8$

$y = -8,$  $\theta = \tan^{-1}(1) \Longrightarrow \theta = \frac{\pi}{4}$

$$\theta = 45° + 180° = 225°$$

Or, $\theta = \frac{\pi}{4} + \pi$ ,  $\theta = 5\frac{\pi}{4}$

Roots: $i, (1 \pm \sqrt{3})/2$

**d)** $8^{\frac{1}{8}}$

$z = 8^{\frac{1}{6}} \Rightarrow z^6 - 8 = 0,$ $\boxed{8^{\frac{1}{6}} = 0.1666}$ using scientific calculator

$\left.\begin{array}{l} x = 8 \\ y = 0 \end{array}\right\} \Rightarrow r = 8\ and\ \boxed{\theta = 0}$

$n = 6,\quad \boxed{k = 0,1,2,3,4,5,6,7}$

$w_0 = z^{\frac{1}{6}} = 1\left[\cos\left(\frac{0+2(0)\pi}{6}\right) + \text{i}\sin\left(\frac{0+2(0)\pi}{6}\right)\right] = 1.414[1 + 0i] \qquad w_0 = 1.414$

$w_1 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+2\pi}{6}}_{=60°}\right) + \text{i}\sin\left(\frac{0+2\pi}{6}\right)\right] = 1.414[0.5 + 0.866i],$

$w_1 = 0.707 + 1.225i$

$w_2 = z^{\frac{1}{6}} = 1\left[\left(\underbrace{\frac{0+4\pi}{6}}_{=120°}\right) + i\,sin\left(\frac{0+4\pi}{6}\right)\right] = 1.414[-0.5 + 0.866i],$

$w_2 = -0.707 + 1.225i$

$w_3 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+6\pi}{6}}_{=\pi}\right) + i\,sin\left(\frac{0+6\pi}{6}\right)\right] = 1.414[-1 + 0i], \qquad w_3 = -1.414$

$w_4 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+8\pi}{6}}_{/\ *180=240°}\right) + i\,sin\left(\frac{0+8\pi}{6}\right)\right] = 1.414[-0.5 - 0.866i], \qquad w_4 = -0.707 - 1.225i$

$w_5 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+10\pi}{6}}_{\frac{5}{3}\pi=300°}\right) + i\,sin\left(\frac{0+10\pi}{6}\right)\right] = 1.414[0.5 - 0.866i], \qquad w_5 = 0.707 - 1.225i$

$w_6 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+12\pi}{6}}_{2\pi=360°}\right) + i\,sin\left(\frac{0+12\pi}{6}\right)\right] = 1.414[1 + 0i],$

$w_6 = 1.414$

$w_7 = z^{\frac{1}{6}} = 1\left[\cos\left(\underbrace{\frac{0+14\pi}{6}}_{=420°}\right) + \text{i}\sin\left(\frac{0+14\pi}{6}\right)\right]$

$= 1.414[0.5 + 0.866i],$

***Roots are then,***

$$\pm\sqrt{2}, \frac{(1 \pm i\sqrt{3})}{\sqrt{2}}, \frac{(-1 \pm i\sqrt{3})}{\sqrt{2}}$$

## *Examples*

**4.** Find a value of $\arg z$ when $z_1 \neq 0$ and $z_2 \neq 0$;

**a)** $z = \frac{z_1}{z_2}$ **b)** $z = z^n (n = 1,2, \dots)$ **c)** $z = z^{-1}$

***Solution :***

**a)** $Z = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta}}{2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$

Since, $\boxed{\arg(z) = \arg(z_1) - \arg(z_2)}$

**b)** $z = z^n (n = 1,2, \dots)$

$z = r^n e^{in\theta} = r^n[\cos(n\theta) + \sin(n\theta)] \Rightarrow \arg z^n \Rightarrow n \arg z_1$

**c)** $z = z^{-1} \Rightarrow r^{-1} e^{-i\theta_1}, \; \arg z^{-1} = -\arg z_1$

**5.** Find four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficient.

$$z^4 + 4 = 0 \Rightarrow (z^2 + 2i)(z^2 - 2i) = 0$$

$$z^2 + 2i = 0 \Rightarrow z^2 = -2i, \qquad z = (-2i)^{\frac{1}{2}}$$

$or, \; z^2 - 2i = 0 \Rightarrow z^2 = 2i, \; z = (2i)^{\frac{1}{2}}$

$$\left.\begin{aligned} z &= (-2i)^{\frac{1}{2}} \\ z &= (2i)^{\frac{1}{2}} \end{aligned}\right\} \quad \text{Compute the solution!}$$

Roots

Or, $z^4 + 4 = 0$

$$z^4 + 2 + 2z + 2 - 2z = 0$$
$$z^4 + 2z + 2 + 0 - 2z + 2 = 0$$
$$z^4 + 2z + z^2 + 2 - z^2 + 0 - 2z + 2 = 0$$

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$(z^2 + 2z + 2)(z^2 - 2z + 2) = 0$

$(z^2 + 2z + 2)$ or $(z^2 - 2z + 2) = 0$ *then complete the solution*.

**_Example:_**

Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$, Find (in the form $x + iy$):

- **a)** $z_1 z_2$
- **b)** $(z_1 + z_2)^2$,
- **c)** $1/z_1$,
- **d)** $z_1/z_2$.

**_Solution_**:

$$z_1 z_2 = (2 + 3i)(4 - 5i) = 8 + 2i + 15 = 23 + 2i$$

$$(z_1 z_2)^2 = (23 + 2i)^2 = 23^2 + 2(23)(2)i - 4 = \cdots$$

$z_1 = 2 + 3i$

$\underline{z_2 = 4 - 5i}$

$z_1 + z_2 = 6 - 2i$

$(z_1 + z_2)^2 = (6 - 2i)^2 = 36 - 24i - 4 = 32 - 24i$

$$\frac{1}{} \quad \frac{}{2 + 3i} * \frac{2 - 3i}{2 - 3i} = \frac{2 - 3i}{4 + 9} = \frac{2 - 3i}{16} = \frac{1}{8} - \frac{3}{16} i$$

$$\frac{z_1}{} \quad \frac{}{4 - 5i} * \frac{4 + 5i}{4 + 5i} = \frac{(2 + 3i)(4 + 5i)}{} \quad \frac{}{16 + 25} = \frac{-7 + 22i}{41}$$

$$= -\frac{7}{41} + \frac{22}{41} i$$

المحاضرة السادسة :

التطبيقات الفيزيائية للاعداد والدوال العقدية .

# Physical Applications

## 1. Wave Equation

The mathematical representation of wave is given by $y = Ae^{i(kx-wt)}$ , Find the intensity $I$ .

***Solution:***

$z = re^{i\theta}$ $\Longrightarrow$

$$z \equiv y$$
$$r \equiv A = \text{amplitude}$$
$$\theta \equiv angle\ (\ or\ phase.) \equiv (kx - wt)$$
$$In\ PHYSICS$$
$$\theta\ \ is\ called\ (\boldsymbol{PHASE})$$

$since,\ \boxed{zz^* = |z|^2 = r^2}\ \ in\ \boldsymbol{mathematics}$

$\boxed{yy^* \equiv y^- = |y|^2 = A^2}\ \ in\ \boldsymbol{physics}$

i.e. $yy^* = |y|^2 = Ae^{\boldsymbol{i}(kx-wt)} * Ae^{-\boldsymbol{i}(kx-wt)}$

Since ,

$$Ae^{\boldsymbol{i}(kx-\omega t)} * Ae^{-\boldsymbol{i}(kx-wt)} = \boldsymbol{A^2 e^{i(kx-\omega t)-i(kx-\omega t)}} = \boldsymbol{A^2 e^0}$$

$$|y|^2 = \boldsymbol{A^2} * (\boldsymbol{1}) = \boldsymbol{A^2}$$

$$\Longrightarrow |y|^2 \equiv \quad \boldsymbol{I = A^2}$$

***Activity :*** Find $y_1 + y_2, y_1 - y_2, y_1 y_2, |y_1 - y_2|$ for wave of the same phase of different amplitude.

***Answer*** **:**

$\boldsymbol{y_1 + y_2}$

$$y \ = A_1 e^{\boldsymbol{i}(k_1x_1 - \omega_1 t_1)}$$

$$y \ = A_2 e^{\boldsymbol{i}(k_2x_2 - \omega_2 t_2)}$$

$$y_1 + y_2 = A_1 \quad {}^{(k_1x_1 - \omega_1 t_1)} + A_2 e^{\boldsymbol{i}(k_2x_2 - \omega_2 t_2)}$$

for wave of the same phase of different amplitude ,

$$(k_1x_1 - \omega_1 t_1) \equiv (k_2x_2 - \omega_2 t_2) = (kx - \omega t)$$

Then ,

$$y_1 = A_1 e^{\boldsymbol{i}(kx - \omega t)}$$

$$y_2 = A_2 e^{\boldsymbol{i}(kx - \omega t)}$$

$$y_1 + y_2 = (A_1 + A_2)\ [cos(kx - \omega t) + \boldsymbol{i}\ sin\ (kx - \omega t)]$$

$$y_1 + y_2 = (A_1 + A_2)\ [cos(kx - \omega t) + \boldsymbol{i}\ sin\ (kx - \omega t)]$$

$$|y_1 + y_2| = (A_1 + A_2)$$

For, $\boldsymbol{y_1 - y_2}$

$$y_1 - y_2 = (A_1 - A_2)\ [cos\left(kx - \omega t\right) + \boldsymbol{i}sin\ \left(kx - \omega t\right)]$$

Now,

$\boldsymbol{|y_1 - y_2|}$

$$|y_1 - y_2| = \sqrt{(A_1 - A_2)^2 cos^2(kx - \omega t) + (A_1 - A_2)^2 sin^2(kx - \omega t)}$$

$$|y_1 - y_2| = \sqrt{(A_1 - A_2)^2 \{cos^2(kx - \omega t) + sin^2(kx - \omega t)\}}$$

Then ,

then ,*Capacitive* ,

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(50)(20)} F \quad , \quad C = \frac{10^6}{2\pi(50)(20)} \mu F = 159.2\ \mu F$$

(d) *Impedence* , $Z$=$15\angle -60^o$

$$Z = 15\angle -60^o = 15[\cos(60^o) + j\sin(-60^o)] = 7.50 - j12.99\Omega.$$

Hence,

resistance =$7.50\Omega$ and capacitivie reactance , $X_C = 12.99\Omega$.

Since

$$\frac{1}{2\pi f C} \quad \text{then capacitive, } C = \frac{1}{2\pi f X_C} = \frac{10^6}{2\pi(50)(12.99)} \mu\text{F} = 245\mu\text{F}$$

***Example*** :

An alternative voltage of 240V, 50Hz is connected across an impedance of $(60 - j100)\Omega$ .

Determine the :

(a) resistance
(b) capacitance
(c) magnitude of the impedance and its phase angle and
(d) current flowing

Solution

Impedance $Z = (60 - j100)\Omega$. Hence resistance = 60Ω.

(a) Capacitive reactance $X_C = 100\Omega$ , and Since

$$X_C = \frac{1}{2\pi f C}$$

then *capacitance* ,

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(50)(100)} = \frac{10^6}{2\pi(50)(100)} \mu\text{F} = 31.83\ \mu\text{F}$$

(c ) *Magnitude of impedance* ,

$$|Z| = \sqrt{(60)^2 + (-100)^2} = 116.6\ \mu\text{F}$$

***Phase angle***, or $\arg Z = tan^{-1}\left(\frac{-100}{60}\right) = -59.04^o$

(b) *Current flowing* ,

$$I = \frac{V}{Z} = \frac{240\angle 0^o}{116.6\angle -59.04^o} = 2.058\angle 59.04^o \text{ A}$$

***Example*** :

For the parallel circuit shown in Fig., determine (,using complex numbers)the values of:

(a) current $I$ ,and
(b) its phase relative to the 240V supply

Solution

- ✓ Current $I = \frac{V}{Z}$ .
- ✓ Impedance $Z$ for the three-branch parallel circuit is given by :

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

Where

$$Z_1 = 4 + j3\,,\quad Z_2 = 10 \text{ and } \quad Z_3 = 12 - j5$$

✓ Admittance , $Y_1 = \frac{1}{Z_1} = \frac{1}{4+j3}$

$$= \frac{1}{4 + j3} \times \frac{\mathbf{4 - j3}}{\mathbf{4 - j3}} = \frac{4 - j3}{4^2 + 3^2}$$

$$Y_1 = 0.160 - j0.120 \text{ siemens}$$

✓ Admittance , $Y_2 = \frac{1}{Z_2} = \frac{1}{10}$

$$Y_2 = 0.10 \text{ siemens}$$

✓ Admittance , $Y_3 = \frac{1}{3} = \frac{1}{12-j5}$

$$= \frac{1}{12-j5} \times \frac{\mathbf{12+j5}}{\mathbf{12+j5}} = \frac{12+j5}{12^2+5^2}$$

$$Y_3 = 0.0710 + j0.0296 \text{ siemens}$$

**Total admittance,** $\boldsymbol{Y} = \boldsymbol{Y_1} + \boldsymbol{Y_2} + \boldsymbol{Y_3}$

$= (0.160 - j0.120) + (0.10) + (0.0710 + j0.0296)$

$= 0.331 - j0.0904$

$\boldsymbol{Y} = 0.343\angle - 15.28^o$ siemens

Current , $I = \frac{V}{Z} = VY$

$I = (240\angle 0^o)\ (0.343\angle - 15.28^o) = 82.32\angle - 15.28^o\ A$

$\boldsymbol{I = 82.32\angle - 15.28^o\ A}$

## *3.Mechanics*

(a) ***Three coplanar forces***

Determine the

- magnitude and
- direction of the resultant of the three coplanar forces given below ,when they act at a point :
  - Force *A*, 10N acting at $45^o$ from the poisitive horizontal axis ,
  - Force *B* , 8N acting at $120^o$from the positive horizontal axis ,
  - Force *C* , 15N acting at $210^o$from the positive horizontal axis.

Solution

The space daigram is shown in the figure .

The forces may be written as complex numbers. Thus

**force** $\boldsymbol{A}$ **,** $\boldsymbol{F_A = 10\angle 210^o}$ ,

**force** $\boldsymbol{B}$ **,** $\boldsymbol{F_B = 8\angle 120^o}$ ,

**force** $\boldsymbol{C}$ **,** $\boldsymbol{F_C = 15\angle 210^o}$.

- The resultant force $= f_A + f_B + f_C$

$= 10\angle 45^o + 8\angle 120^o + 15\angle 210^o$

$\boldsymbol{F} = 10(cos45^o + jsin\ 45^o) + 8(cos120^o + jsin\ 120^o) + 15(cos210^o + jsin\ 210^o)$F

$= 7.071 + j7.071) + (-4.00 + j6.928) + (-12.99 - j7.5) = -9.919 + j6.499$

$\boldsymbol{F} = -9.919 + j6.499$

Since

$-9.919 + j6.499$ lies in the second quadrant

❖ *Magnitude of resultant force*

$$F = \sqrt{(-9.919)^2 + 6.499^2} = 11.86N$$

❖ *Direction of resultant force*

$$= tan^{-1}\left(\frac{6.499}{-9.919}\right) = 146.77^o$$

since $-9.919 + j6.499$ ) lies in the second quadrant).

## *Solving the harmonic oscillator*

A harmonic oscillator is governed by the equation

$$ma = -kx \qquad ....(1)$$

Where $a$ is acceleration .

This provides us with the differential equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad ... (1)$$

We know that the solution of this equation can be written as sine and cosine :

$$x(t) = Acos(\omega_0 t) + Bsin\,(\omega_0 t)$$

Where $\omega_0 = \sqrt{k/m}$ , and the constants $A$ and $B$ are chosen to match initial conditions.

For example if $x(0) = x_0$ and $v(0) = \left(\frac{dx}{dt}\right)\Big|_{t=0}$ , then

$$A = x_0 \quad ,B = 0$$

An equivalent way to write this solution is to put

$$x = Ae^{i(\omega_0 t + \phi_0)})$$

And match initial conditons by adjusting the constants $A$ and $\phi_0$ . With the initial conditions chosen above .we would put

.

$$A = x_0 \text{ and } \phi_0 = 0$$

This is easy to see:

$$\frac{dx}{dt} = i\omega_0 A e^{i(\omega_0 t + \phi_0)} = i\omega_0 x \qquad \dots (2)$$

$$\frac{}{dt^2} = -\omega^2 A e^{i(\omega_0 t + \phi_0)} = -\omega^2 x \qquad \dots (3)$$

Substitute eq. (3), (2) in to (1)

$$-\omega^2 m x + kx = 0$$

Then the solution satisfied

$$\omega_0 = \sqrt{k/m}$$

# Complex Numbers

2022

## *Complex Functions*

**المحاضرة السابعة :**

**الدوال العقدية مع أمثلة تطبيقية**

## Complex Functions الدوال العقدية

Real part of f(z) — Imaginary part of f(z)

$$w = f(z) = U(x, y) + iV(x, y)$$

The variable $z$ is sometimes called an *independent* variable, $w$ is called a *dependent* variable.

**Example:**

If $f(z) = z^2$, then find $f(z)$ if $z = 2i$

$$f(2i) = (2i)^2 = -4$$

**Example**

If $w = z^2$, then $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, and the transformation is

$$f(z) = z^2 = (x + iy)^2$$
$$f(z) = x^2 + 2xyi - y^2$$

$$U = x^2 - y^2, \qquad V = 2xy$$

**Example:**

Illustrate the function

$u(x,y)$ $v(x,y)$

$$\Rightarrow u(x, y) = \sqrt{x^2 + y^2}$$
$$\Rightarrow v(x, y) = -y$$

## *Elementary Functions*

**1. Polynomial function are defined by**

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z)$$

Where $a_0 \neq 0, a_1, \dots, a_n$ are complex constants and $n$ is a positive integer called the degree of the polynomial $P(z)$.

*Complex Exponential function*

$e \quad , i \quad , \pi$

**$Euler's formula$**

$$e^{i\theta} = cos\theta + isin\theta$$

***De'Moiver theorem***

$$e^{i(n\theta)} = e^{in\theta} = cos(n\theta) + isin(n\theta)$$

**2. Rational Algebraic functions are defined by** $w = \frac{P(z)}{Q(z)}$

Where $P(z), Q(z)$ are polynomials $Q(z) \neq 0$.

**3. Exponential functions are defined by**

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Where $e = 2.71828 \ldots$ is the natural base of logarithms .If $a$ is real and positive, we define: $\boxed{a^z = e^{z \ln a}}$ ( ***prove***)

*Where* $ln\, a$ *is the natural logarithm of* $a$.

***Proof***

$$w = a^z \Longrightarrow ln\, w = ln\, a^z \Longrightarrow ln\, w = z \ln a$$

$$\therefore w = e^{z \ln a}$$

* دالة اسية مرفوعة الى ( الاس مضروب في لوغاريتم الاساس )

Complex exponential functions have properties similar to those of real exponential function. For example,

$$e^{z_1} * e^{z_2} = e^{z_1+z_2},$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

**4. Trigonometric functions** (or circular functions) $\cos z, \sin z$, etc. can be defined in terms of exponential functions as follows.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{}{i\left(e^{iz} + e^{-iz}\right)}$$

- $\sin^2 z + \cos^2 z = 1, \quad 1 + \tan^2 z = \sec^2 z,$
  $1 + \cot^2 z = \csc^2 z$
- $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$
  $\tan(-z) = -\tan z$
  $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
  $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

## 5. Hyperbolic Functions are defined as follows

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \qquad \operatorname{csch} z = \frac{\cosh z}{\sinh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

The following properties hold

- $\cosh^2 z - \sinh^2 z = 1, \quad 1 - \tan^2 z = \operatorname{sech}^2 z,$
  $\coth^2 z - 1 = \operatorname{csch}^2 z$
- $\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z,$
  $\tanh(-z) = -\tanh z$
- $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$
  $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$
- The following relations exist between the trigonometric or circular functions and the hyperbolic functions.

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z,$$
$$\tan iz = i \tanh z$$
$$\sinh iz = i \sin z, \cosh iz = \cos z,$$
$$\tanh iz = i \tan z$$

## Logarithmic Functions:

If $z = e^w$, then we write $w = ln\, z$, called the natural logarithm of $z$

$$w = \ln z, \qquad z = re^{i\theta}$$

$$w = ln(re^{i\theta}) = ln\, r + ln\, e^{i\theta}$$

$$\boxed{w = \boldsymbol{lnz} = \mathbf{ln}\, \boldsymbol{r} + \boldsymbol{i(\theta + 2k\pi)}}, \qquad k = 0, \pm1, \pm2, \ldots$$

Where $z = re^{i} = re^{i(\theta+2k\pi)}$

$ln\, z \Longrightarrow$ is multiple-valued function. ***The principle branch*** of principle value of $ln\, z$ is defined as:

$$ln\, r + i\theta, \qquad \text{where } 0 \leq \theta \leq 2\pi$$

- The logarithmic function can be defined for real bases other than $e$. If

$$\boxed{z = a^w}, \qquad w = \log_a z$$

Where $a > 0$ and $a \neq 0,1$.

In this case $\boxed{z = e^{w \ln a}}$ and so $\boxed{w = (ln\, z)\, /\, ln\, a)}$

## 6. Inverse Trigonometric Functions ( تعطى )

If $\boxed{z = \sin w}$ , the $w = \sin^{-1} z$ is called the inverse sine of $z$: or arc sine of $z$.

$$w = \sin^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1 - z^2}\right)$, • $\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2 - 1}\right)$

## 7. Inverse Hyperbolic functions( تعطى )

If $= \sinh w$ , the $w = \sinh^{-1} z$ is called the inverse hyperbolic sine of $z$.

$$w = \sinh^{-1} z \Longrightarrow \text{(multiple-valued function)}$$

- $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$, $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

## *Example*

**1.** Prove that:

**a)** $e^{z_1} * e^{z_2} = e^{z_1 + z_2}$

**b)** $|e^z| = e^x$

**c)** $e^{z+2k\pi i} = e^z, \; k = 0, \pm1, \pm2, \ldots$

***Solution***

**a)** By definition

$$e^z = e^x(\cos y + i \sin y) \qquad \text{where} \begin{cases} z = x + iy \\ z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{cases}$$

$\therefore e^z \ * e^z \ = e^{x_1}(\cos y_1 + i\sin y_1) * e^{x_2}(\cos y_2 + i\sin y_2) = e^{x_1} * e^{x_2}$
$(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) = e^{x_1+x_2}\cos(y_1 + y_2$
$) + i\, e^{x_1+x_2} sin(y_1 + y_2)$

**b)** $|e^z| = |e^x(\cos y + i\sin y)| = |e^x||\cos y + i\sin y|$

**c)** By part (a)

$$e^{z+2k\pi i} = e^z * e^{2k\pi i} = e^z(\cos 2k\pi + i\sin 2k\pi) = e^z$$

$2k\pi i \Longrightarrow$ is a ***period*** of the function ,In particular, $\boldsymbol{e^z}$ ***has*** **period** $\boldsymbol{2\pi i}$.

**2. Find the value of $\boldsymbol{ln(1-i)}$. What is the principle value**?

Or, determine the Principle value of $ln(1-i)$.

$$ln\, z = ln\left(re^{i\theta}\right) = ln\, r + i\theta$$

$z = 1 - i \Longrightarrow x = 1, y = -1,$

$$r = \sqrt{2}$$

$$\theta = -45°, \text{or} -\frac{\pi}{4}$$

$\ln z = \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right) =$

$\frac{1}{2} ln\, 2 + \frac{7\pi}{4} i + 2k\pi i$

The ***principle value*** is $(\boldsymbol{k = 0})$, $\left(\boldsymbol{\frac{1}{2}\ln 2 + \frac{7\pi}{4} i}\right)$

**a)** Find the principle value of $i^i$.

***Solution***

**(a)** by definition $z^i = e^{i\ln z} = e^{i\{\ln r + i(\theta+2k\pi)\}} = e^{i\ln r - (\theta + 2k\pi)}$

$$= e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$$

$0 \le \theta \le 2\pi$

$$k = 0, \text{then } z^i = e^{-\theta}\{\cos(\ln r) + i\sin(\ln r)\}$$

**(b)** By definition, $i^i = e^{i\ln i}$, the principle value $= i^i = e^{-\pi/2}$

Now ,

Let $w = Z^Z$ , $ln\, w = ZlnZ$

---

**Example :** $(1+i)^{(2-5i)}$

**Method#1**

**Let** , $w=(1+i)^{(2-5i)}$

$lnw=(2-5i)\,ln(1+i)$ , $\quad ln\,z=lnr+i\theta$ ,

$x=1\,,y=1\,,r=\sqrt{2}\,,\quad \theta=45^0\,,or\quad \frac{\pi}{4}$

$ln(1+i)=ln\sqrt{2}+i\frac{\pi}{4}$

$lnw=(2-5i)\,ln(1+i)=(2-5i)[ln\sqrt{2}+i\frac{\pi}{4}]$

$lnw=2ln\sqrt{2}+i\frac{\pi}{2}-5ln\sqrt{2}\,i+\frac{5\pi}{4}$

Then taking the exponential of both sides ……

**Method#2**

$$w=(1+i)^{(2-5i)}$$

$$w=e^{(2-5i)ln(1+i)}=e^{(2-5i)[ln\sqrt{2}+i\frac{\pi}{4}]}=e^{2ln\sqrt{2}+i\frac{\pi}{2}-5ln\sqrt{2}\,i+5\frac{\pi}{4}}$$

$$w=e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)+i(\frac{\pi}{2}-5ln\sqrt{2})}$$

$$w=e^{\left(2ln\sqrt{2}+5\frac{\pi}{4}\right)}\times e^{i(\frac{\pi}{2}-5ln\sqrt{2})}$$

Make use of Euler's formula

$$e^{i\theta}=cos\theta+isin\theta$$

**Solve** : $\quad i^{(2i)}$ , (1 − [redacted])$^{(2.5-1.3i)}$ , $10^{-7i}$ , $(-9+\sqrt{3}\,i)^{(-6i)}$

(a) $\quad w=i^{2i}$

$\quad lnw=(2i)ln(i)$

$$ln\,z=lnr+i\theta$$

$$ln\,i=ln1+i\frac{\pi}{2}=0+i\frac{\pi}{2}\;,\quad ln\,i=i\frac{\pi}{2}$$

Then ,

$$lnw = (2i)\left[i\frac{\pi}{2}\right] = -\pi$$

$$ln\,w = -\pi$$

*Taking* ***exponential*** *for both sides:*

$$e^{lnw} = e^{-\pi} \rightarrow \quad w = e^{-\pi}$$

$$w = e^{-\pi}$$

*(b)* $(1-3i)^{(2.5-1.3i)}$

$$w = (1-3i)^{(2.5-1.3i)} \quad \rightarrow ln\,w = ln\left[(1-3i)^{(2.5-1.3i)}\right]$$

$$ln\,w = (2.5-1.35i)ln(1-3i)$$

$$ln\,z = lnr + i\theta$$

$\boldsymbol{ln(1-3i)}$ !!!!!!

$x = 1\,, \quad y = -3\,, r = 3.16, \quad \theta = -7.1.6^{o} \;\; or\,, \theta = 288.4^{o} \;\;, \theta = 1.6\pi$

$$ln(1-3i) = ln(3.16) + 288.4^{o}\,\boldsymbol{i}$$

$$\therefore \quad ln\,w = ln(3.16) + 288.4^{o}\,\boldsymbol{i} = 1.15 + 288.4^{o}\,i$$

Then ,

$$e^{lnw} = e^{1.15+288.4^{o}\,i}$$

$$w = e^{1.15} \times e^{288.4^{o}\,i}$$

$$(1-3i)^{(2.5-1.3i)} = 3.16[cos288.4^{o} + isin288.4^{o}]$$

$$(1-3i)^{(2.5-1.3i)} = (3.16\times 0.316) - \boldsymbol{i}(3.16\times 0.949)$$

$$(1-3i)^{(2.5-1.3)} = 1.14 - 2.99\boldsymbol{i}$$

## *Complex Functions*

## أمثلة ( أضافية ) عن الدوال العقدية

**Example 1 : Find the value of $ln(1-i)$. What is the principle value?**

Or, determine the Principle value of $ln(1-i)$.

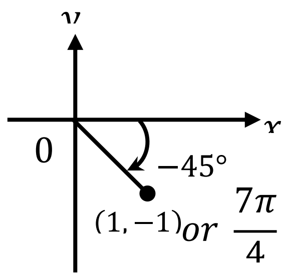
$$ln\,z = ln(re^{i\theta}) = ln\,r + i\theta \qquad \boldsymbol{\pi = 3.14}$$

$$z = 1 - i \implies x = 1, \quad y = -1, \ r = \sqrt{2}$$

$\theta = -45°$, or $\left(-\frac{\pi}{4}\right)$ , $or\ (+\frac{7\pi}{4})$, $or$ , $(-0.785\ \ or\ \ 5.49)$

$$ln\,z = \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right)$$

$$= \frac{1}{2} ln\,2 + \frac{7\pi}{4} i + 2k\pi i$$




The ***principle value*** is $(\boldsymbol{k = 0})$, $\left(\frac{1}{2}\boldsymbol{ln\,2} + \frac{7\pi}{4}\boldsymbol{i}\right)$

Or ,

$$ln(1-i) = 0.35 - 0.79\boldsymbol{i} \quad , \qquad or\ \ , ln(1-i) = 0.35 + 5.49\boldsymbol{i}$$

**Example 2:** Find the value of $e^{1+i}$.

***Solution***

$$e^{1+i} = e^1 . e^i$$

باستخدام الحاسبة اليدوية ، فان

$$e^1 = 2.718$$

أما $e^i$ ، والتي تمثل **دالة اسية عقدية** يتم استخدام صيغة أويلر (Euler's formula )

$$e^{i\theta} = cos\theta + isin\theta$$

$$e^i = \cos(1) + isin(1) = 0.54 + 0.84\boldsymbol{i}$$

**Example 3:**

سبق وان تم الاشارة في محاضرات سابقة بان هناك :

- ✓ **Power of** $Z \Rightarrow \quad Z^n$ , $n$ is integer number may be positive or negative
- ✓ **Root of** $Z \Rightarrow \; Z^{1/n}$ , $n$ is integer and positive .

الان اذا كان ( عدد عقدي او دالة عقدية) مرفوع ( لعدد عقدي او دالة عقدية ) أخر ى ... (سبق وان اعطينا الامثلة التالية ) :

$\boldsymbol{Z^Z}$ , $\boldsymbol{(1+i)^i, i^{2i}, 5^{-(1-2i)}}$ , ... ..

**Example 3 :** $(1+i)^{(2-5i)}$

**Method#1**

**Let** , $w = (1+i)^{(2-5i)}$

$$lnw = (2-5i)\,ln(1+i) \qquad \ldots(1)$$

$$\ln z = lnr + i\theta \;,$$

$$x = 1 \,, y = 1 \;,$$

$$r = \sqrt{x^2 + y^2} \;\Rightarrow r$$ [redacted] ,

$\theta = tan^{-1}(y/x) \quad \Rightarrow \theta = 45^0$ , $or \ \frac{\pi}{4}$ , $or\ 0.785\ rad.$

$$ln(1+i) = ln\sqrt{2} + i\frac{\pi}{4}$$

Or ,

$$ln(1+i) = 0.346 + 0.785\boldsymbol{i}$$

وبتعويض هذا الناتج في المعادلة (1) نجد ان

$$lnw = (2-5i)\,ln(1+i) = (2-5i)[ln\sqrt{2} + i\frac{\pi}{4}]$$

$$lnw = 2ln\sqrt{2} + i\frac{\pi}{2} - 5ln\sqrt{2}\,i + \frac{5\pi}{4}$$

Then taking the exponential of both sides ......

**Method#2**

$$w = (1+i)^{(2-5i)}$$

$$w = e^{(2-5i)\,ln(1+i)} \qquad ...(1)$$

$$w = e^{(2-5i)[ln\sqrt{2}+\boldsymbol{i}\frac{\pi}{4}]} = e^{(2-5i)[0.346+0.785\boldsymbol{i}]}$$

$$w = e^{0.692} \times e^{1.57\boldsymbol{i}} \times e^{-1.73\boldsymbol{i}} \times e^{3.925}$$

$$w = e^{4.617} \times e^{-0.16\boldsymbol{i}}$$

باستخدام الحاسبة اليدوية لانها دالة حقيقية

تحسب باستخدام صيغة اويلر ( Euler's formula ) لانها دالة اسية عقدية ،

$$\boldsymbol{e^{i\theta} = cos\theta + isin\theta}$$

$$w = 101.19[cos0.16 - \boldsymbol{i}sin0.16]$$

راجع الملاحظة السابقة في طريقة ايجادها ،

$$w = 99.9 - 16.13\boldsymbol{i}$$

*Example 4*: Evaluate $sin(i)$

***Solution***

***Method#1:***

$$sin\,z = \frac{e^{iz} - e^{-iz}}{2i}$$

بالمقارنة مع المسالة نجد ان $z = i$

$$\therefore \; sin\,z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{i}{2i^2}[e^{iz} - e^{-iz}]$$

بضرب البسط والمقام ب ( $i$ )

Then ,

$$sin\,z == \frac{-i}{2}[e^{iz} - e^{-iz}]$$

$$\therefore \ sin(i) = \left(\frac{-i}{2}\right)\left[e^{i(i)} - e^{-i(i)}\right] = \left(\frac{-i}{2}\right)\left[e^{-1} - e^{+1}\right]$$

( لاحظ ان $e^{+1}$ ، $e^{-1}$ دوال اسية حقيقية يمكن ايجاد قيمتها باستخدام الحاسبة اليدوية ) .

$$\boldsymbol{e^{+1}} = 2.718\,,$$

$$\boldsymbol{e^{-1}} = \frac{1}{e^{1}} = \frac{1}{2.718} = \mathbf{0.367}$$

$$\therefore \ sin(i) = \left(\frac{-i}{2}\right)\left[e^{i(i)} - e^{-i(i)}\right] = \left(\frac{-i}{2}\right)[0.367 - 2.718] = 1.18\boldsymbol{i}$$

$$\therefore \ sin(i) = 1.18\boldsymbol{i}$$

**Example 5**: Evaluate $sinh(1+i)$

**Solution**

$$sinhz = \frac{e^{Z} - e^{-Z}}{2}$$

$$sinh(1+i) = \frac{e^{(1+i)} - e^{-(1+i)}}{2} = \frac{1}{2}\left[e^{1} \times e^{i} - e^{-1} \times e^{-i}\right]$$

$$sin\,h(1+i) = \frac{1}{2}[2.718(0.54 + 0.84i) - 0.367(0.54 - 0.89i)] = 0.635 + 1.30i$$

$$sin\,h(1+i) = 0.635 + 1.29i$$

Now,

$$isin\,h(1+i) = -1.29 + 0.635\boldsymbol{i}$$

## *Activity* :

1. Evaluate $cos(i)$
2. Evaluate $cos(1+i)$
3. Evaluate $cos\,h(i)$
4. Evaluate $cosh(1+i)$

## Complex Derivative

## المحاضرة الثامنــة :

## مشتقة الدوال العقدية وتطبيقاتها

Derivative of complex function were defined in terms of *limit* and the other in terms of *formulas* ;

أي لغرض ايجاد مشتقة الدالة العقدية فان هناك طريقتين :

- ✓ الاولى هي استخدام (الغايات) و
- ✓ الثانية استخدام ( قواعدالمشتقات )

و هو نفس الاسلوب المتبع سابقا في ايجاد مشتقة الدوال في حسابان التكامل و التفاضل للدوال الحقيقية وكالاتي :

$$f'(Z_0) = \frac{\Delta w}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{f(Z_0 + \Delta Z) - f(Z_0)}{\Delta Z}$$

Or,

| | |
|---|---|
| $\frac{dw}{dZ} = f'(z)$ | **$1^{st}$** derivative of $w$ with respect to z |
| $\frac{d^2w}{dZ^2} = \frac{d}{dz}(\frac{dw}{z}) = f''(z)$ | **$2^{nd}$** derivative of $w$ with respect to z |
| $\frac{d^nw}{dZ^n} == f^{(n)}(z)$ | **$n^{th}$** derivative of $w$ with respect to z |

## *Rules for differentation*

If $f(z), g(z)$ and $h(z)$ are " *analytic* " function of $z$ the following differentation rules are *valid* :

# Complex Derivative



| | |
|---|---|
| **1.** | $\frac{d}{dz}[f(z)+g(z)] = \frac{d}{dz}f(z) + \frac{d}{dz}g(z)$ <br> $= f'(z) + g'(z)$ |
| **2.** | $\frac{d}{dz}[f(z)-g(z)] = \frac{d}{dz}f(z) - \frac{d}{dz}g(z)$ <br> $= f'(z) - g'(z)$ |
| | $\frac{d}{dz}\{cf(z)\} = c\frac{d}{dz}f(z)$ , $and$ $c$ is constant |
| **4.** | $\frac{d}{dz}[f(z)g(z)] = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$ <br> $= f(z)g'(z) + g(z)f'(z)$ |
| | $\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$ <br> $= \frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$ , if $g(z) \neq 0$ |

والجدول التالي يوجز مشتقات اهم الدوال :

| | |
|---|---|
| $\frac{d}{dz}z^a = az^{a-1}$ | $\frac{d}{dz}z^z = z^z(1+\ln z)$ |
| $\frac{d}{dz}a^z = a^z\ln(a)$ | $\frac{d}{dz}\ln z = \frac{1}{z}$ |
| $\frac{d}{dz}e^z = e^z$ | $\frac{d}{dz}\log_a z = \frac{1}{z\ln a}$ |
| $\frac{d}{dz}\sin(z) = \cos(z)$ | $\frac{d}{dz}\sinh(z) = \cosh(z)$ |
| $\frac{d}{dz}\cos(z) = -\sin(z)$ | $\frac{d}{dz}\cosh(z) = \sinh(z)$ |
| $\frac{d}{dz}\tan(z) = \sec^2(z)$ | $\frac{d}{dz}\tanh(z) = 1-\tanh^2(z) = \text{sech}^2(z)$ |
| $\frac{d}{dz}\cot(z) = -\csc^2(z)$ | $\frac{d}{dz}\coth(z) = -\text{csch}^2(z)$ |
| $\frac{d}{dz}\csc(z) = -\csc(z)\cot(z)$ | $\frac{d}{dz}\text{csch}(z) = -\text{csch}(z)\coth(z)$ |
| $\frac{d}{dz}\sec(z) = \sec(z)\tan(z)$ | $\frac{d}{dz}\text{sech}(z) = -\text{sech}(z)\tanh(z)$ |
| $\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$ | $\frac{d}{dz}\sinh^{-1}(z) = \frac{1}{\sqrt{1+z^2}}$ |
| $\frac{d}{dz}\cos^{-1}(z) = -\frac{1}{\sqrt{1-z^2}}$ | $\frac{d}{dz}\cosh^{-1}(z) = \frac{1}{\sqrt{z+1}\sqrt{z-1}}$ |
| $\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$ | $\frac{d}{dz}\tanh^{-1}(z) = \frac{1}{1-z^2}$ |
| $\frac{d}{dz}\cot^{-1}(z) = -\frac{1}{1+z^2}$ | $\frac{d}{dz}\coth^{-1}(z) = \frac{1}{1-z^2}$ |
| $\frac{d}{dz}\csc^{-1}(z) = -\frac{1}{z\sqrt{z^2-1}}$ | $\frac{d}{dz}\text{csch}^{-1}(z) = -\frac{1}{z^2\sqrt{1+\frac{1}{z^2}}}$ |
| $\frac{d}{dz}\sec^{-1}(z) = \frac{1}{z\sqrt{z^2-1}}$ | $\frac{d}{dz}\text{sech}^{-1}(z) = -\frac{1}{z(z+1)\sqrt{\frac{1-z}{1+z}}}$ |

**Example 1** : Find the 1st derivative for the function

$$f(z) = [z + (z^2+1)^2]^2$$

$$\frac{d}{dz}[z + (z^2+1)^2]^2 = 2[z + (z^2+1)^2]\frac{d}{dz}[z + (z^2+1)^2]$$
$$= 2[z + (z^2+1)^2][1 + 2(z^2+1) \times 2z]$$

At $z = 1 + i$ , then

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$$\frac{d}{dz}[z+(z^2+1)^2]^2$$
$$= 2[(1+i)+\{(1+i)^2+1\}^2]\times[1+2\{(1+i)^2+1)\times 2(i)\}]$$
$$= \dots\dots$$

**Example 2** : Find the 1st derivative for the function

$$f(z) = \left[\frac{(z+2i)(i-z)}{(2z-1)}\right] \ ,at \quad z = i$$

***Solution***

$$(z+2i)(i-z) = zi - z^2 - 2 - 2iz = -2 - iz - z^2$$

$$\frac{d}{dz}\left[\frac{(z+2i)(i-z)}{(2z-1)}\right] = \frac{d}{dz}\left[\frac{-2-iz-z^2}{(2z-1)}\right] = \frac{(2z-1)(-i-2z)-(-2-iz-z^2)(2)}{(2z-1)^2}$$

**Example 3** : Find the 1st derivative for the function $[z+(z^2+1)^2]^2$

$$\frac{d}{dz}[z+(z^2+1)^2]^2 = 2\{z+(z^2+1)^2\} * [1+2(z^2+1)(2z)]$$

**Example 4** : Find the 1st derivative for the function

$$f(z) = (z+2\sqrt{z})^{1/3}$$

$$\frac{d}{dz}(z+2\sqrt{z})^{1/3} = \frac{1}{3}[(z+2\sqrt{z})^{-2/3}] * \left(1+2\left(\frac{1}{2}\right)z^{-1/2}\right] = \dots\dots\dots$$

**Example 5** : Find $\frac{d}{dz}(1+z^2)^{3/2}$

$$\frac{d}{dz}(1+z^2)^{\frac{3}{2}} = \frac{3}{2}(1+z^2)^{1/2} \times \frac{d}{dz}(1+z^2)$$
$$\frac{d}{dz}(1+z^2)^{3/2} = \frac{3}{2}(1+z^2)^{1/2} \times (2z) = \cdots \dots$$

**Example 6** : Find $\frac{d}{dz}\left[\frac{(z+2i)(i-z^2}{(2z-1)^2}\right] at \quad z = i$

**Example 7** : Find $1^{st}$ derivative for the function $z^{lnz}$

Let , $w = z^{lnz}$

$$lnw = (lnz) \times ln(z) \rightarrow \ lnw = (ln\,z)^2$$

$$\frac{1}{w}\frac{dw}{dz} = 2ln\,z\left(\frac{1}{z}\right) \rightarrow \frac{dw}{dz} = w \times 2ln\,z \times \left(\frac{1}{z}\right)$$

$$\therefore \ \frac{dw}{dz} = z^{lnz} \times 2lnz \times z^{-1}$$

$$\frac{dw}{dz} = 2(z^{lnz-1})lnz$$

**Example 8** : Find $\frac{dw}{dz}$ for the function $z^i$

Let , $w = z^i$

$$lnw = ln(z^i) \rightarrow \ lnw = i(lnz)$$

$$\frac{1}{w}\frac{dw}{dz} = i\left(\frac{1}{z}\right) \rightarrow \frac{dw}{dz} = i\left(\frac{1}{z}\right) * w$$

$$\therefore \ \frac{dw}{dz} = w \times \frac{i}{z} = \left(\frac{i}{z}\right)(z^i) = iz^{i-1}$$

**Example 9** : Find $\frac{dw}{dz}$ $\{sin(iz-2)^{tan^{-1}(z+3i)}\}$

Let , $w = sin(iz -$ [redacted]$)^{tan^{-1}(z+3i)}$

$$lnw = tan^{-1}(z+3i) \times ln\{sin(iz-2)\}$$

$\frac{1}{w}\frac{dw}{dz} = tan^{-1}(z+3i) \times \frac{1}{sin(iz-2)} \times cos(iz-2) \times i + ln\{sin(iz-2) \times \frac{1}{1+(z+3i)^2} \times (1)\}$

$$\frac{1}{w}\frac{dw}{dz} = w \times \dots\dots$$

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**Example 10** : Find $\frac{d}{dz}\, ln[z-\frac{3}{2}+\sqrt{z^2-3z+2i}\,]$

$$\frac{d}{dz}ln\left[z-\frac{3}{2}+\sqrt{z^2-3z+2i}\right]=\frac{1}{z-\frac{3}{2}+\sqrt{z^2-3z+2i}}\;\frac{d}{dz}[z-\frac{3}{2}+\sqrt{z^2-3z+2i}]$$

$$=\frac{1}{z-\frac{3}{2}+\sqrt{z^2-3z+2i}}\;[1+\frac{1}{2}(z^2-3z+2i)^{-1/2}(2z-3)]$$

**Example 11** : Find $\frac{d}{dz}\, tan^{-1}(z+3i)^{-1/2}$

$$\frac{d}{dz}\,tan^{-1}(z+3i)^{-1/2}=\frac{1}{1+[(z+3i)^{-1/2}]^2}\times\frac{d}{dz}(z+3i)^{-1/2}$$

$$=\frac{1}{1+\left(\frac{1}{\sqrt{z+3i}}\right)^2}(-\frac{1}{2})(z+3i)^{-\frac{3}{2}}\times(1)$$

**Example 12** : Find $\frac{d}{dz}\, cos^{-1}(sinz-cosz)$

$$\frac{d}{dz}\,cos^{-1}(sinz-cosz)=\frac{-1}{\sqrt{1-(sinz-cosz)^2}}\,\frac{d}{dz}\,(sinz-cosz)$$

**Example 13** : Find $\frac{d}{dz}\, sinh(z+1)^2$

$$\frac{d}{dz}sinh(z+1)^2=cosh(z+1)^2\times 2(z+1)(1)$$

**Example 14** : Find $\frac{d}{dz}\, cosh^{-1}(lnz)$

$$\frac{d}{dz}\,cosh^{-1}(lnz)=-\frac{1}{\sqrt{1-(lnz)^2}}\times\frac{1}{z}$$

**Activity** : Find $\frac{d}{dz}\,5^{(z^2+i)}$ , *Hint* $\frac{d}{dz}\,a^z=a^z\times lna\times\frac{d}{dz}(z)$

ثم قارن مع مثال7 ، 8 و 9 ماذا تستنتج !!!

$$\frac{d}{dz}\,5^{(z^2+i)}=5^{(z^2+i)}\times ln5\times\frac{d}{dz}(z^2+i)=5^{(z^2+i)}\times ln5\times(2z)$$

# *Complex Function Separation*

**المحاضرة 9**

## تجزئـــة الدوال العقدية الى اجزائها الحقيقية و الخيالية

Complex function $w = f(z) = U(x,y) + \boldsymbol{i}\, V(x,y)$

***Example 1:*** Separate each of the following functions into real and imaginary parts , *or in other words*

***Find*** :

$U(x,y)$ *and* $V(x,y)$ such that $f(z) = U + \boldsymbol{i}\, V$

**(a)** $f(z) = z$

**(b)** $f(z) = z^2 - i$

**(c)** $f(z) = 2z^2 - 3iz$

**(d)** $f(z) = z + \frac{1}{z}$

**(e)** $f(z) = \frac{z^2+1}{z}$

**(f)** $f(z) = \frac{1-z}{1+z}$

**(g)** $f(z) = \sqrt{z}$

**(h)** $f(z) = lnz$

**(i)** $f(z) = e^{3iz}$

**(j)** $f(z) = sin2z$

**(k)** $f(z) = cosz$

**(l)** $f(z) = z^z$

**(m)** $f(z) = z^2 e^{2z}$

**(n)** $f(z) = sinhz$

***Solution***

***(a)*** $f(z) = z$

$$f(z) = x + iy \rightarrow u(x,y) = x \ , v(x,y) = y$$

***(b)*** $f(z) = z^2 - i$

$$f(z) = (x + iy)^2 - i$$

$$f(z) = x^2 + 2xyi - y^2 - i$$

# Complex Function Separation

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$$f(z) = (x^2 - y^2) + i(2xy - 1)$$

**$U(x,y)$** **$V(x,y)$**

*(c)* $f(z) = 2z^2 - 3iz$

$$f(z) = 2(x+iy)^2 - 3i(x+iy)$$

$$f(z) = 2[x^2 + 2xyi - y^2] - 3xi + 3y$$

$$f(z) = 2x^2 + 4xyi - 2y^2 - 3xi + 3y$$

$$f(z) = (2x^2 - 2y^2 - 3y) + \boldsymbol{i}\,(4xy - 3x)$$

**$U(x,y)$** **$V(x,y)$**

*(d)* $f(z) = z + \frac{1}{z}$ *the same as* $(\boldsymbol{e})$ $f(z) = \frac{z^2+1}{z}$

***Method#1***

$$f(z) = (x+iy) + \frac{1}{(x+iy)} = (x+iy) + \frac{1}{(x+iy)} \times \frac{x-iy}{x-iy}$$

$$f(z) = z + \frac{1}{z} = x + iy + \frac{x-iy}{x^2+y^2}$$

$$f(z) = x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$$

$$f(x+iy) = \left[\frac{x(x^2+y^2)+x}{x^2+y^2}\right] + i\left[\frac{y(x^2+y^2)-y}{x^2+y^2}\right]$$

**$U(x,y)$** **$V(x,y)$**

***Method#2***

$$f(z) = (x+iy) + \frac{1}{(x+iy)} = \frac{(x+iy)^2+1}{\phantom{x+iy}} \quad \frac{}{x+iy}$$

$$z + \frac{1}{z} = \frac{(x^2-y^2+1)+2xyi}{x+iy} \times \frac{x-iy}{x-iy}$$

$$f(z) =$$

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***Method#3***

This problem can be sovled with the aid of *complex conjugate* of $z$ , i.e.

$f(z) = z + \frac{1}{z}$

$$f(z) = \frac{z^2+1}{z} \quad \rightarrow \quad f(z) = \frac{z^2+1}{z} \times \frac{\bar{z}}{\bar{z}}$$

$$f(z) = \frac{z^2\bar{z} + \bar{z}}{z\bar{z}} = \frac{z^2\bar{z} + \bar{z}}{|z|^2}$$

Then substitute $z = x + iy$ in above equation ...what about the result. Dísucss!!.

*(f)* $f(z) = \frac{1-z}{1+z}$

$$f(x+iy) = \frac{1-(x+iy)}{1+(x+iy)} = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{(1-x)(1+x) - iy(1+x) - iy(1-x) - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - iy - xyi - iy + xyi - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - 2iy - y^2}{(1+x)^2 + y^2}$$

$$f(z) = \frac{1-x^2-y^2}{(1+x)^2+y^2} - \boldsymbol{i}\frac{2y}{(1+x)^2+y^2}$$

$\boldsymbol{U(x,y)}$ $\boldsymbol{V(x,y)}$

*(g)* $f(z) = \sqrt{z}$

$$f(x+iy) = (x+iy)^{1/2}$$

Or in *polar form* , since $z = re^{i\theta} \rightarrow \quad z = r[cos\theta + isin\theta]$

Then ,

$$z^{1/n} = r^{1/n}[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)]$$

$k = 0$ , $n = 2$ , $\quad r = \sqrt{x^2 + y^2}$ , $\quad \theta = tan^{-1}(y/x)$

$$\therefore \quad \sqrt{z} = z^{1/2} = (x^2 + y^2)^{1/2}[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)]$$

*Where*

$$x = r\cos\theta \quad , \quad \cos\theta = \frac{x}{r}$$

$$y = r\sin\theta \quad , \quad \sin\theta = \frac{y}{r}$$

$$\therefore \quad \sqrt{z} = z^{1/2} = r^{1/2}\left[\frac{x}{2r} + i\,\frac{y}{2r}\right] = \frac{1}{2\sqrt{r}}[x + iy]$$

$$\therefore \quad z^{1/2} = \frac{1}{2(x^2 + y^2)^{1/2}}[x + iy] = \frac{x}{2(x^2 + y^2)^{1/2}} + i\,\frac{y}{2(x^2 + y^2)^{1/2}}$$

$\boldsymbol{U(x,y)}$ $\quad$ $\boldsymbol{V(x,y)}$

**(h)** $\quad f(z) = lnz$

Since , $\quad lnz = lnr + i\theta$

$$U(x,y) = lnr$$

$$V(x,y) = \theta$$

Since

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = tan^{-1}(y/x)$$

$$lnz = ln(x^2 + y^2)^{1/2} + \boldsymbol{i}\, tan^{-1}(y/x)$$

$$lnz = \frac{1}{2}ln((x^2 + y^2) + \boldsymbol{i}\, tan^{-1}(y/x)$$

$\therefore \quad U(x,y) = \frac{1}{2}ln((x^2 + y^2)$ $\quad$ and $\quad$ $V(x,y) = tan^{-1}(y/x)$

**(i)** $\boldsymbol{f(z) = e^{3iz}}$

$$e^{3iz} = e^{3i(x+iy)} = e^{3ix-3y} = e^{3ix} \times e^{-3y}$$

$$= e^{-3y}[\cos(3x) + \boldsymbol{i}\sin(3x)]$$

$$\therefore \quad e^{3iz} = e^{-3y}\cos(3x) + \boldsymbol{i}\, e^{-3y} sin(3x)]$$

$$\therefore \quad U(x,y) = e^{-3y}\, cos(3x) \quad \& \quad V(x,y) = e^{-3y} sin(3x)$$

**(j)** $f(z) = sin(2z)$

$$f(x+iy) = sin\, 2(x+iy) = \sin(2x+2iy)$$

$$= sin(\,2x)\cosh(2y) + \boldsymbol{i}\, cos(2x)\ \ sin(2y)$$

$$sin(2z) = sin(\,2x)\, cosh(2y) + cos(2x)\, sin\,h(2y)$$

Where ,

$$sin(iz) = \boldsymbol{i}sinh(z)$$

**(k)** $f(z) = sin(2z)$

***Method#1***

Let , $f(z) = cosz$

$$f(x+iy) = \cos(x+iy)$$

باستخدام جيب تمام حاصل جمع زاويتين ، فان

$$\mathrm{Cos}(x+iy) = cosxcos(iy) - sinxsin(iy)$$

ذكرنا سابقا في موضوع الدوال العقدية ، ان هناك علاقة تربط بين الدوال المثلثية ( جيب وجيب تمام ) بالدوال الزائدية ، اي

$$sin(iy) = isinhy$$
$$cos(iy) = coshy$$

***Method#2***

$$f(z) = cosz$$

كما هو معلوم،

$$cosz = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

وبالتعويض عن $z = x + iy$ بالمعادلة اعلاه ، نحصل على ،

$$cosz = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\left[e^{ix}.e^{-y} + e^{-ix}.e^{y}\right]$$

$$=\frac{1}{2}[e^{-y}(cosx+isinx)+e^{y}(cosx-isinx)]$$

$$=\frac{1}{2}[\boldsymbol{e^{-y}cosx}+ie^{-y}sinx+\boldsymbol{e^{y}cosx}-ie^{y}sinx]$$

الان نجمع الاجزاء الحقيقية مع بعضها والخيالية مع بعضها ، فتصبح ،

$$cosz=\frac{1}{2}[(e^{-y}\boldsymbol{cosx}+e^{y}\boldsymbol{cosx})+i(e^{-y}\boldsymbol{sinx}-e^{y}\boldsymbol{sinx})]$$

ناخذ عامل مشترك بين الاجزاء الحقيقية و الاجزاء الخيالية ،نحصل على

$$cosz=\frac{1}{2}[cosx(e^{y}+e^{-y})-isinx\,(e^{y}-e^{-y})]$$

بما ان ،

$$sinhz=\frac{1}{2}(e^{y}-e^{-y})$$

$$coshz=\frac{1}{2}(e^{y}+e^{-y})$$

وعليه فان ، $cosz=cosxcoshy-\boldsymbol{i}\,sinxsinhy$

**(l)** $f(z)=z^{z}$

Let , $w=z^{z}$

$$lnw=ln(z^{z})=zln(z)$$

$$w=e^{zlnz}$$

Then ,

$$w=e^{(x+iy)\ln(x+iy)}$$

*Since ,*

$$ln(z)=lnr+i\theta$$

*Or,*

$$ln(z)=ln\sqrt{x^{2}+y^{2}}+\boldsymbol{i}tan^{-1}(y/x)$$

$$w=e^{(x+iy)[\frac{1}{2}ln(x^{2}+y^{2})+\boldsymbol{i}\,tan^{-1}(y/x)}$$

$$\therefore \quad w = e^{\frac{1}{2}x\, ln\left(x^2+y^2\right)-y\, tan^{-1}(y/x)} \times e^{\boldsymbol{i}\, [\frac{y}{2} ln\left(x^2+y^2\right)+x\, tan^{-1}(y/x)}$$

**(m)** $f(z) = z^2 e^{2z}$

$$f(x+iy) = (x+iy)^2 e^{2(x+iy)}$$

$$= (x^2 - y^2 + 2ixy)e^{2x} * \; e^{2yi}$$

$$= e^{2x}[(x^2 - y^2 + 2ixy)] \times [\cos(2y) + isin(2y)]$$

$$z^2 e^{2z} = e^{2x}\left(x^2 - y^2\right)\cos(2y) + \boldsymbol{i}\; e^{2x}\left(x^2 - y^2\right)sin(2y)]$$

$$+\, \boldsymbol{i}\; e^{2x}(2xy)\cos(2y) - e^{2x}(2xy)\sin(2y)$$

# *Analytic (Cauchy –Remann Conditions) and Harmonic complex functions (Laplace Equation)*

## المحاضرة 10

✓ الدوال العقدية التحليلية و (شروط كوشي- ريمان)

✓ الدوال العقدية التوافقية ( معادلة لابلاس )

*Previous lecture ,*

Complex function $\boldsymbol{w = f(z) = U(x,y) + i\,V(x,y)}$

◈ Complex function to be *analytic* must satify *Cauchy – Remann Conditions* , that is :

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

***Example 1:*** Does the function $f(z) = e^z$ analytic ?.

**Solution**

$$f(z) = e^z = e^{x+iy} = e^x \times e^{iy}$$

$$e^z = e^x[\cos y + i \sin y] = e^x \cos y + i\, e^x \sin y$$

$$\therefore \quad u(x,y) = e^x \cos y \quad , \quad v(x,y) = e^x \sin y$$

*C-R-C's*

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$
$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \; e^x cosy$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *analtyic*

- A complex function to be *harmonic* must satify *Laplace equation* , that is :

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad , or \quad U_{xx} + U_{yy} = 0$$
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad , or \quad V_{xx} + V_{yy} = 0$$

*Or* ,

$$\nabla^2 \psi = 0 \quad where \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The ' *operator* ' $\nabla^2$ is called " *Laplacian*" .

***Example 2 :*** Does the function $f(z) = e^z$ harmonic ?

***Solution***

$$\therefore \quad u(x,y) = e^x cosy \quad , v(x,y) = e^x \, siny$$

$$\frac{\partial U}{\partial x} = e^x cosy \quad , \quad \frac{\partial V}{\partial x} = e^x \, siny$$

$$\frac{\partial^2 U}{\partial x^2} = e^x cosy \quad , \quad \frac{\partial^2 V}{\partial x^2} = e^x \, siny$$
$$\frac{\partial U}{\partial y} = -e^x siny \quad , \quad \frac{\partial V}{\partial y} \; e^x cosy$$
$$\frac{\partial^2 U}{\partial y^2} = -e^x cosy \quad , \quad \frac{\partial^2 V}{\partial y^2} = -e^x \, siny$$

The conditons are ***satisfied*** the the function $f(z) = e^z$ is *harmonic function .*

**Example 3 :** Does the function $f(z) = cosz$ analytic , harmonic ?

**Solution**

$$f(z) = cosz = cosx\ coshy - \boldsymbol{i}\ sinx\ coshy$$

$$\therefore \quad u(x,y) = cosx\ coshy \quad , v(x,y) = -sinx\ coshy$$

$$\frac{\partial U}{\partial x} = -sinx\ coshy \quad , \quad \frac{\partial V}{\partial x} = -cosx\ sinhy$$

$$\frac{\partial^2 U}{\partial x^2} = -cosx\ coshy \quad , \quad \frac{\partial^2 V}{\partial x^2} = sinx\ sinhy$$

$$\frac{\partial U}{\partial y} = cosx\ sinhy \quad , \quad \frac{\partial V}{\partial y} = -sinx\ coshy$$

$$\frac{\partial^2 U}{\partial y^2} = cosx\ coshy \quad , \quad \frac{\partial^2 V}{\partial y^2} = -sinx\ sinhy$$

The conditons are ***satisfied*** the the function $f(z) = cosz$ is *analytic* and *harmonic function .*

**Example 4 :** Does the function $f(z) = lnz$ analytic , harmonic ?

**Note :** $\frac{d}{dz} tan^{-1} z = \frac{-1}{1+z^2}$

**Solution**

$$f(z) = lnz = ln(x + \boldsymbol{i}y)$$

$$lnz = \frac{1}{2} ln(x^2 + y^2) + \boldsymbol{i}\ tan^{-1}(y/x)$$

$$\therefore U(x,y) = \frac{1}{2} l\,n(x^2 + y^2)$$

$$\frac{\partial U}{\partial x} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2} \times \frac{1}{x^2 + y^2} \times 2y = \frac{y}{x^2 + y^2}$$

$$V(x,y) = tan^{-1}(y/x)$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+(y/x)^2} \times (-x^2 y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial V}{\partial y} = \frac{1}{1+(y/x)^2} \times (-x^{-1})$$

***Activity :***

Does the functions

1. $f(z) = z + \bar{z}$ analytic , harmonic ?
2. $f(z) = \frac{1}{z}$ analytic , harmonic ?
3. $f(z) = \frac{z+i}{z-i}$ analytic , harmonic ?
4. $f(z) = zlnz$ analytic , harmonic ?
5. $f(z) = \boldsymbol{i}\, cosz$ analytic , harmonic?

# *Complex Power Series Expansion*

**المحاضرة 11**

## متسلسلات الدوال العقديـــة

- ✓ متسلسلة تايلـــر **( Taylor Series)** ومتسلسلة ماكلورين **( Maclaurin Series )**
- ✓ متسلسلة لورنت **( Laurent Series )**

## *Taylor Series*

Suppose that a function $f$ is analytic throughout a circle $|z - z_0| < R$ , centered at $z_0$ and with radius $R_0$ . Then $f\,(z)$ has the power series representation.

$$f(z) = f(z_0) + f'(z_0)(\,z - z_0\,) + \frac{f''(z_0)}{2!}(\,z - z_0\,)^2 + \cdots . + \frac{f^{(n)}(z_0)}{n!}(\,z - z_0\,)^n + \cdots$$

Where ,

$$n! = n(n-1)(n-2)\ \ldots .3.2.1$$

$$0! = 1\ \ ,\ \ 1! = 1\ \ ,\ \ 2! = 2 * 1 = 2\ \ ,$$

$$3! = 3 * 2 * 1 = 6\,, \ldots .\,etc$$

**ملاحظــة** : عنما تكون $z =$ ■ فان متسلسلة **تايلر** **تؤول** الى متسلسلة **ماكلورين** .

*Here some function expansion and convergent region:*

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < \infty$
2. $sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ \ldots \qquad |z| < \infty$
3. $cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ \ldots \qquad |z| < \infty$
4. $coshz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ \ldots \qquad |z| < \infty$

**5.** $(1+z)^{\propto} = 1 + \propto z - \frac{\propto(\propto-1)z^2}{2!} + \cdots \qquad |z| < 1$

*is the* ***binomial theorem*** *for* $\propto = -1$ ***gives formula 8.***

**6.** $lnz$ ***undefined*** *at* $z = 1$

**7.** $ln(1+z) = z - \frac{z2}{2} + \frac{z^3}{3} - \ldots\ldots \qquad |z| < 1$

**8.** $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad |z| < 1$

**9.** $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$

*The first five expansion are valid for all z , whilst the last three are only valid for* $|z| < 1.$

***Example 1*** Expand $f(z) = ln(1+z)$

*Solution*

$$f(z) = ln(1+z) \quad , \quad f(0) = 0$$
$$f'(z) = \frac{1}{1+z} \quad , \quad f'(0) = 1$$
$$f''(z) = \frac{-1}{(1+z)^2} \quad , \quad f''(0) = -1$$
$$f^{(3)}(z) = \frac{(-1)(-2)}{(1+z)^2} \quad , \quad f^{(3)}(0) = 2$$

..........

..........

$$f^{(n+1)}(z) = \frac{(-1)^n \; n!}{(1+z)^{(n+1)}} \quad , \quad f^{(n+1)}(0) = (-1)^n \; n!$$

$$f(z) = ln(1+z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$f(z) = ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

***Example 2*** Expand $f(z) = ln\frac{(1+z)}{(1-z)}$

*Solution*

$$f(z) = ln\frac{(1+z)}{(1-z)} \quad ,$$

$$f(z) = ln\frac{(1+z)}{(1-z)} = ln(1+z) - ln(1-z)$$

$$ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$$ln(1+z) - ln(1-z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots . + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots . l$$

$$= 2z + \frac{2z^3}{3} + \cdots . = 2(z + \frac{2z^3}{3} + \frac{z^5}{5} \ldots .) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

$$ln\frac{(1+z)}{(1+z)} = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

***Example 3*** Expand $f(z) = \frac{1}{(1+z)^2}$

*Solution*

$$f(z) = \frac{1}{(1+z)^2}$$

$$f(z) = \frac{1}{(1+z)^2} = (1+z)^{-2} \,, \quad f(0) = 1$$

$$f'(z) = \frac{-2}{(1+z)^3} \quad , \quad f'(0) = -2$$

$$f''(z) = \frac{6}{(1+z)^4} \quad , \quad f''(0) = 6$$

$$\therefore \quad \frac{1}{(1+z)^2} = 1 - 2z + 3z^2 + \cdots \qquad provided \quad |z| < 1$$

***Example 4*** Expand $f(z) = \frac{1}{1+z}$

*Solution*

$$f(z) = \frac{1}{1+z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1+z} \quad , f(0) = 1$$

$$f'(z) = \frac{-1}{(1+z)^2} \quad , \quad f'(0) = -1$$

$$f''(z) = \frac{2}{(1+z)^3} \quad , \quad f''(0) = 2$$

...........

...........

$$\therefore \quad \frac{1}{1+z} = 1 - z + z^2 + \cdots$$

**Example 5** If $|z| < 1$ , expand $f(z) = \frac{1}{1-z}$

*Solution*

$$f(z) = \frac{-1}{1-z} = (1+z)^{-1}$$

$$f(z) = \frac{1}{1-z} \quad , f(0) = 1$$

$$f'(z) = \frac{1}{(1-z)^2} \quad , \quad f'(0) = 1$$

$$f''(z) = \frac{2}{(1-z)^3} \quad , \quad f''(0) = 2$$

..........

..........

$$\therefore \quad \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \quad |z| < 1$$

**Example 6** Expand $f(z) = e^{i\theta}$

*Solution*

$$\because \quad \boldsymbol{e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

*Then*

$$e^{i\theta} = 1 + (i\theta) + + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + \boldsymbol{i\theta} - \frac{\theta^2}{2!} - \boldsymbol{i}\frac{\boldsymbol{\theta^3}}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \boldsymbol{i}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\theta^3}}{3!} + \cdots\right)$$

$$e^{i\theta} = cos\theta + isin\theta$$ "*Euler formula*"

**Activity :** $e^{i\pi} + 1 = 0$

*Is called the most beautiful equation in all of mathematics*

- ✓ It is an *identify* that contains the most beautiful entities encountered in math, namely $\boldsymbol{\pi}, \boldsymbol{i}, \boldsymbol{e}, \boldsymbol{0}$ and $\boldsymbol{1}$.
- ✓ It **combines** the real and the imaginary.

# Complex Power Series Expansion

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In mathematics, ***Euler's identity*** (also known as ***Euler's equation***) is the equality

$$e^{i\pi} + 1 = 0$$

where

| | |
|---|---|
| $e$ | is Euler's number, the base of natural logarithms, |
| $i$ | is the imaginary unit, which by definition satisfies $i^2 = -1$, and |
| $\pi$ | is pi, the ratio of the circumference of a circle to its diameter. |

*Euler's* ***identity*** is a special case of *Euler's* ***formula***, which states that for any real number *x*,

$$e^{ix} = \cos x + i\sin x$$

where the inputs of the trigonometric functions sine and cosine are given in ***radians***. In particular, when $x = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

Since

$$cos\,\pi = -1$$

and

$$sin\,\pi = 0$$

it follows that $e^{i\pi} = -1 + 0i$

which *yields Euler's identity* : $e^{i\pi} + 1 = 0$

***Activities***

- $e^0 = 1$
- $e^z \neq 0$
- $|e^z| = e^x \equiv e^{Rez}$
- $arg(e^z) = y + 2n\pi \equiv Imz \;, (n = 0, \mp1, \mp2, \ldots)$

We know that $z' = x' + iy'$ , *then* $\arg(z') = tan^{-1}(\frac{y'}{x'})$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\,cos(y) + ie^x\,sin(y) = x' + iy'$$

$$\arg(e^z) = \tan^{-1}\left(\frac{y'}{x'}\right) = \tan^{-1}\left(\frac{e^i\sin(y)}{e^x\cos(y)}\right) = \tan^{-1}(\tan(y)) = y + 2k\pi$$

$$\Rightarrow \quad arg(e^z) = y + 2\pi k$$

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- $\overline{e^{z}} = e^{\bar{}}$
- $\ln(e^{z}) = z + 2\pi i \quad ,(n = 0, \mp1, \mp2, \dots)$
- $e^{\log z} = z$
- $\ln(z^{1/k}) = \frac{1}{k}\ln z \quad , \quad (k = \mp1, \mp2, \dots)$

## Homework

**1**. Expand the follwing function

(a) $e^{-z} \; ; \; z = 0$

(b) $\cos z \; ; \; z = \pi/2$

(c) $\frac{1}{1+z} \; ; \; z = 1$

(d) $\ln z \; ; \; z = 2$

(e) $z\, e^{2z} \; ; \; z = -1$

**2**.Show that : $\sin z = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots \quad , \; |z| < \infty$

**3**.Show that : $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \quad , \; |z| < 1$

**4**.Show that : $\sec z = 1 + \frac{z^2}{2} + \frac{5}{24} + \cdots \quad , \; |z| < \pi/2$

**5**.Show that : $\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots \quad , \; 0 < |z| < \pi$

**6**. Expand : $\tan^{-1}(iz) = \quad \dots \dots \dots \dots$

## *Laurent's Series*

The method of *Laurent series expansions* is an important tool in complex analysis. Where a *Taylor series* can only be used to describe the *analytic part* of a function, Laurent series allows us to *work around the singularities* of a complex function. To ***do this***, we need to determine ***the singularities*** of the function and can then ***construct several concentric rings*** *with the same center* $z_0$ based on those singularities and obtain a unique Laurent series of $z$-$z_0$ inside each ring where the ***function*** is *analytic*. ***In other words***,

If a function fails to be analytic at a point $z_0$ , one can not apply *Taylor's* theorem **at that point**.Unlike the Taylor series which express $f(z)$ as a series of terms with *non-negative powers* of $z$ , a ***Laurent series*** includes terms with *negative powers*.

### *Theorem*

Suppose that a function $f$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at $z_0$ and let $C$ denote any poisitive orientated simple closed contour around $z_0$ and lying in that domain . Then,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

تمثل $C_n$ محيط الدائرة (الخارجية او الداخلية )التي نصف قطرها $R_1$ و $R_2$ وكلاهما بالاتجاه الموجب وعكس عقرب الساعة وعلى التوالي .

**ملاحظـة :**

1. يسمى الجزء $c_0 + c_1(z - z_0) + +c_2(z - z_0)^2 + \cdots$ بالجزء التحليلي من متسلسلة لورنت والجزء الثاني $\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$ بالجزء الاساسي .
2. اذا كان الجزء الاساس ( الرئيسي ) من متسلسلة لورنت ( صفرا) فان متسلسلة لورنت تصبح " متسلسلة تايلر ".

***Example 6*** If $|z| > 1$ , expand $f(z) = \frac{1}{1-z}$ using Laurent series

***Ans.*** $\frac{1}{1-z} = -\sum_{n=1}^{\infty} \frac{1}{n}$ , $|z| > 1$

$$f(z) = \frac{1}{1-z} = \frac{1}{z\left(\frac{1}{z}-1\right)} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\left(\frac{1}{z}\right)\left(\frac{1}{\left(1-\frac{1}{z}\right)}\right)$$

$$\frac{1}{1-z} = -\left(\frac{1}{z}\right)[1 + \frac{1}{z} + \frac{1}{z^2} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots.$$

$$\frac{1}{1-z} = -[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots.]$$

Here $f(z) = \frac{1}{1-z}$ is ana;lytic **everywhere** apart from the **singularity** at $z =$ ■.

Above are the expansions of $f$ in the regions inside and outside the circle of radius 1, centered on $z = 0$ , where $|z| < 1$ is the region *inside* the circle and $|z| > 1$ is the region *outside* the circle.

***Example 7*** Expand $\frac{e^{2z}}{(z-1)^3}$ ; $z = 1$

*Solution*

Let , $u = z - 1 \rightarrow z = 1 + u$ , $2z = 2(1 + u)$

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3} \times e^{2u} = \frac{e^2}{u^3}[1 + 2u + \frac{(2u)^2}{2!} + \cdots.$$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \cdots.$$

حيث ان $z = 1$ هو قطب ( pole ) من الرتبة الثالثة .

**Example 8** Expand $\frac{1}{z^2(z-3)^2}$ ; $z=3$

*Solution*

Let , $u=z-3 \rightarrow z=u+3$ , $z=u+3$

$$\frac{1}{z^2(z-3)^2}=\frac{1}{u^2(3+u)^2}=\frac{1}{9u^2(1+u/3)^2}$$

$$=\frac{1}{9u^2}\{1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^2+\cdots\}$$

$$\therefore \quad \frac{1}{z^2(z-3)^2}=\frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4u}{24^3}+\cdots$$

$$\frac{1}{z^2(z-3)^2}=\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4u}{24^3}+\cdots$$

حيث ان $z=3$ هو قطب ( pole ) من الرتبة الثانية .

**Example 9** Let $f(z)=\frac{1}{2+z}$ .Determine the Laurent series around $z=1$

**Solution**

Obviously, we have a simple pole at $z=-2$. Hence, we are dealing with a radius of **3** and want to find the Laurent series for both $|z-1|<3$ *and* $|z-1|>3$. The Laurent series will reduce to a Taylor series inside $|z-1|<3$ where $f(z)$ is analytic.

For $|z-1|<3$, we refer to the well-known geometric series. We begin by trying to create a $(z-1)$ term in the denominator.

$$f(z)=\frac{1}{2+z}=\frac{1}{2+z-1+1}=\frac{1}{3+(z-1)}=\left(\frac{1}{3}\right)\frac{1}{1-\left(\frac{-(z-1)}{3}\right)}$$

Since $\left|\frac{-(z-1)}{3}\right|<1$, we can now represent the function as a series:

$$\Rightarrow f(z)=\frac{1}{3}\sum_{n=0}^{\infty}\frac{-(z-1)^n}{3^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}} \quad for \quad |z-1|<3$$

Which is just a Taylor series as the function is analytic inside the region. For $|z-1|>3$, we can use $\frac{3}{|z-1|}$ and follow our previous work to obtain:

$$f(z)=\frac{1}{3+(z-1)}=\frac{1}{z-1}\frac{1}{1-\left(\frac{-3}{z-1}\right)}=\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n3^n}{(z-1)^n}$$

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*In this case* ,we have obtained the Laurent expansion.The generalized residue for the outer ring $|z-1|>3$ is the coefficient of $\frac{1}{z-1}$ , that is $b_1=1$ .

**Example 10** Expand $f(z)=\frac{1}{(z+1)(z+3)}$ as Laurent series when:

*(a)* $1<|z|<3$

*(b)* $|z|<3$

*(c)* $0<|z+1|<2$

*(d)* $|z|<1$

**Solution**

Using partial fractions

$$\frac{1}{(z+1)(z+3)}=\frac{A}{z+1}+\frac{B}{z-3}$$

$$1=Az+3A+Bz+B$$

$$A+B=0$$

$$3A+B=1$$

$$A=-B$$

$$3A-A=1\rightarrow \quad \boldsymbol{A=1/2}$$

$$\boldsymbol{B=-1/2}$$

$$\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\frac{1}{z+1}\right)-\frac{1}{2}\left(\frac{1}{z+3}\right)$$

*(a)* $|z|>1$ *, then*

$$\frac{1}{2}\left(\frac{1}{z+1}\right)=\frac{1}{2z(1+\frac{1}{z})}=\frac{1}{2z}[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots]$$

$$=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots$$

*(b)* $|z|<3$ *, then*

$$\frac{1}{2}\left(\frac{1}{z+3}\right)=\frac{1}{6(1+\frac{z}{3})}=\frac{1}{6}[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots]$$

$$\frac{1}{2}\left(\frac{1}{z+3}\right)=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\cdots$$

وعليه فان مفكوك لورنت صحيح لكل من $|z|>1$ , $|z|<3$ اي ان $|z|>3$ ينتج بالطرح وهو :

$$=\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\cdots$$

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(c) $0 < |z+1| < 2$

Let , $u = 1 + z$ , then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

$$0 < |z+1| < 2 \quad or\ , \quad |u| < 2 \ , \quad u \neq 0$$

(d) *If* $|z| < 1$ *, then*

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots$$

$$= \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

اذا كان $|z| < 3$ ، نجد من الفرع ( $a$ )

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

وبالتالي فان مفكوك لورنت المطلوب والصحيح لكل من $|z| < 1$ ، $|z| < 3$ ، اي ان $|z| < 1$ ينتج بالطرح :

$$= \frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

وهي متسلسلة تايلر .

***Example 11!!*** Determine the Laurent series for $f(z) = \frac{1}{(z+5)}$ that are valid in the regions**:**

*(a)* $|z| < 5$

*(b)* $|z| > 5$

***NOTE*** :

$$f(z) = \frac{1}{5(1+\frac{z}{5})} = \cdots. \qquad , \qquad f(z) = \frac{1}{z\left(1+\frac{5}{z}\right)} = \cdots.$$

***Example 12!!*** Determine the Laurent series for $f(z) = \frac{1}{(z-i)^2}$ **;** $z_0 = i$

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## *Homework*

***1***. Expand the follwing function

(a) $\frac{1}{\sqrt{1+z^3}}$ ; $z=1$

(b) ) $\frac{1}{\sqrt{1+z^3}}$ ; $z=0$

$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{\ }$ $\qquad (2)(4)(6)$ $z^9$

(c) $sin^{-1}z =$

$-\frac{3}{3} + \frac{(1)(3)}{(2)(4)}\frac{z^5}{5} + \frac{(1)(3)(5)}{(2)(4)(6)}\frac{z^7}{7}$ ; $|z| < 1$

(d) $f(z) = \ln(3 - iz), f(0) = ln3$

(e)) $f(z) = \frac{1}{z-3}$ as Laurent series for

(a) $|z| < 3$

(b) $|z| > 3$

*Ans.*

(a) $\frac{-1}{z} - \frac{z}{9} - \frac{z^2}{9} - \frac{z^3}{81} + ..$

(b) $\frac{-1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \ldots.$

*2*.Expand :

$\frac{z}{(z-1)(2-z)}$ as Laurent series for ,

(a) $|z| < 1$ , (b) $1 < |z| < 2$ , (c) $|z| > 2$ , (d) $|z-1| < 1$ , (e) $0 < |z-2| < 1$

***Ans.***

(a) $\frac{-1}{2}z - \frac{3z^2}{4} - \frac{7z}{8} - \frac{15z^3}{16} - \ldots.$

(b) $\frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \ldots.$

(c) $\frac{-1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} + \cdots.$

(d) $\frac{-1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} + \ldots.$

(e) $1 - \frac{2}{(z-2)} - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} + \ldots.$

*3*.Expand : $f(z) = \frac{1}{z(z-2)}$ ; $0 < |z| < 2$ , $|z| > 2$

*4*.Expand : $f(z) = \frac{z}{1+z^2}$ ; , $|z-3| > 2$

*5*.Expand : $f(z) = \frac{1}{(z-2)^2}$ ; $|z| < 2$ , $|z| > 2$

*6*.Expand the following functions as Laurent series around $z = 0$

(a) $f(z) = \frac{1-cosz}{z}$ , [ *Ans.* $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} \ldots$

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(b) $f(z) = \frac{e^{z^2}}{z^3}$ , [ *Ans.* $\frac{1}{z^3} - \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \frac{z^5}{4!} + \frac{z^7}{7!} + \cdots$

(c) $f(z) = zsinh\sqrt{z}$ , $(d)\ f(z) = ze^z$

*7*. State the singular points for the functions :

*(a)* $\frac{1}{2(sinz-1)^2}$ , $(b)\ \frac{z}{(e^{1/z}-1)^2}$ , $(c)\cos(z^2 - z^{-2})$ , $(d)\frac{z}{(e^{1/z}-1)}$

$(e)\ tan^{-1}(z^2 + 2z + 2)$ , *(f)* $(\frac{z}{(e^z-1)}$

# *Complex Integrals*

## المحاضرة 12

✓ التكاملات الخطية العقدية ( Complex line integral )

Liouvalli theorem (a)

Morera's theorem (b)

✓ تكاملات كوشي (Cauchy's integral )

From " Calculus " , the integration say for example

- $\int_0^1 f(x)dx$ , $dx$ *with be varied from* $x_1 = 0$ , $x_2 = 1$ .
- $\int_2^3 f(y)dy$ , $dy$ *with be varied from* $y_1 = 2$ , $y_2 = 3$ .
- $\int_{1,2}^{3,4} (3xy + y^2)dx + (2x^2y - yx + x^2)dy$ ,

هنا التكامل يتضمن دوال معتمدة على $x$ و $y$ . هذه جميعا " تكاملات خطية " ولابد ان توجد علاقة تربط $y$ مع $x$ ، على سبيل المثال :

1. $y = 2x$ *striaght line equation*
2. $y = x^2$ *parabola equation*

ففي الحالة الاولى $y = 2x$ يتم اعطاء قيم $y$ مباشرة او قيمة حيث نعوض عن او بما يساويها فتتحول المعادلة بدلالة بعد واحد ، أما :

$$dy = 2dx$$

$$dy = 2xdx$$

الحالة الثانية ، يتم تحديد " نقطتان " في السوال ضمن المعطيات ومن خلال القيمتين يتم ايجاد قيمة أما $y$ أو $x$ عن طريقة معادلة الخط المستقيم و ابجاد الميل وكالاتي :

Along the striagt line joining the points $(1,2) \rightarrow (3,4)$

بحيث يكون $C$ هو المستقيم الواصل بين النقطتين $(1,2) \rightarrow (3,4)$ ، أي

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$$slope = m = \frac{y_2 - y_1}{\quad} \qquad \frac{}{3-1} = \frac{2}{2} = 1$$

$$m = 1$$

من معادلة ( ميل + نقطة ) ،

$$y_2 - y_1 = m(x_2 - x_1)$$

$$y - 2 = (1)(x - 1)$$

$$\therefore \quad y = x + 1 \; \rightarrow dy = dx$$

## Complex line integral التكاملات الخطية العقدية

*Previous lecture ,*

Complex function $\quad w = f(z) = U(x,y) + \boldsymbol{i}\, V(x,y)$

Since , $\quad z = x + iy$

Let , $\qquad \Delta z = \Delta x + i\Delta y \quad$ ,or $\qquad dz = dx + idy$

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$\therefore \quad \int_C f(z)dz = \int_C udx - \int_C vdy + \boldsymbol{i} \int_C udy + \boldsymbol{i} \int_C vdx$$

Real part of line integral ($\int_C udx - \int_C vdy$)

Imaginary part of line integral ($\int_C udy + \int_C vdx$)

***Example 1*** Evaluate the integral $\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz \;\; where \;\; y = 2x^2$

*Solution*

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz = \int_{(1,1)}^{(2,8)} (x^2 - iy^2)(dx + idy)$$

$$= \int_{(1,1)}^{(2,8)} x^2dx + \int_{(1,1)}^{(2,8)} y^2dy + \boldsymbol{i} \int_{(1,1)}^{(2,8)} x^2dy - \boldsymbol{i} \int_{(1,1)}^{(2,8)} y^2dx$$

$$\because \quad y = 2x^2 \quad \rightarrow dy = 4xdx$$

$$y^2 = 4x^4$$

Substitute in integration equation

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz \quad = \int_1^2 x^2dx + \int_1^2 16x^5dx + \boldsymbol{i} \int_1^2 4x^3dx - \boldsymbol{i} \int_1^2 4x^4dx$$

$$= \frac{x^3}{3}\bigg|_1^2 + \frac{16x^6}{6}\bigg|_1^2 + \boldsymbol{i}x^4|_1^2 - \boldsymbol{i}\frac{4x^5}{3}\bigg|_1^2$$

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz \quad = \left(\frac{8}{3} - \frac{1}{3}\right) + \left(\frac{1024}{6} - \frac{16}{6}\right) + i(16 - 10 - i\left(\frac{128}{5} - \frac{4}{5}\right)$$

$$\int_{(1,1)}^{(2,8)} (x^2 - iy^2)dz \quad = \frac{511}{3} + \boldsymbol{i}\frac{49}{5}$$

**ملاحظة** : يمكن التعبير عن المتغيرين $x$ و $y$ كدوال بمتغير واحد لنقل $t$ والذي يسمى " وسيطا " بين المتغير $x$ و $y$ كما موضح في المثال التالي .

***Example 2*** Evaluate the integral $\int_C z^2dz$ where C is striaght line joining the points : (0,0) → (2,1) ,where $C: \quad Z(t) = x(t) + iy(t)$

*Solution*

$$y - y_1 = m(x - x_1) \qquad \text{"straight line equation"}$$

$$slope = m = \frac{y_2 - y_1}{} \qquad \frac{}{2 - 0} = \frac{1}{2}$$

$$m = \frac{1}{2}$$

$$y - 0 = \frac{1}{2}(x - 0)$$

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$$\therefore \quad y = \frac{1}{2}x \;\rightarrow x = 2y$$

ومن الواضح ان $x$ ، $y$ هما دالتين للمتغير $t$ ( حسب ماورد في السوال ) وعليه يمكن ان نفترض :

Let , $y = t \quad \rightarrow x = 2t$ then $0 \leq t \leq 1$

$$\therefore \quad C \quad : \quad z(t) = 2t + it \qquad 0 \leq t \leq 1$$

$$dz = 2dt + idt$$

*Or* ,

$$dz = (2 + i)dt$$

Now ,

$$\int_C z^2 dz = \int_0^1 (2t + it)^2\,(2 + i)dx$$

$$= (2 + i)\int_0^1 (4t^2 + 4t^2 i - t^2)dt$$

$$= (2 + i)\int_0^1 (3t^2 + i4t^2)dt$$

$$\int_C z^2 dz = (2 + i)\left[t^3 + \frac{4}{3}it^3\right]\Big|_0^1 = (2 + i)(1 + i\frac{4}{3})$$

$$\int_C z^2 dz = \frac{2}{3} + \boldsymbol{i}\frac{11}{3}$$

***Activity*** :

If we choose $y = \frac{t}{2}$ , $x = t$ … *continue*

***Example 3*** Evaluate the integral $\int_C z^2 dz$ where path equation $C$ is
$Z(t) = t + it^2 \qquad 0 \leq t \leq 1$

*Solution*

$$Z(t) = t + it^2$$

$$\therefore \quad dz = dt + \boldsymbol{i}\,2tdt$$

$$\int_C z^2 dz = \int_0^1 (t+it^2)(1+2it)dt \quad , \quad 0 \le t \le 1$$

$$= \frac{(t+it^2)^3}{3}\Big|^{1} = \frac{1}{3}(1+i)^3 = \frac{-2}{3} + \frac{2}{3}\boldsymbol{i}$$

$$\int_C z^2 dz = \frac{-2}{3} + \frac{2}{3}\boldsymbol{i}$$

## نظرية كوشي ــ كورسات Cauchy-Goursat theorem

**The Cauchy–Goursat Theorem**

Suppose that a function $f$ is analytic in a simply connected domain $D$. Then for every simple closed contour $C$ in $D$,

$$\oint_C f(z)\, dz = 0.$$

**لتكن $f(z)$ دالة تحليلية في منطقة ما $\mathcal{R}$ وكذلك على حدودها $C$ او اذا كانت $f(z)$ دالة تحليلية في كل نقطة على وداخل منحني مغلق بسيط $C$ فان :**

$$\oint_C f(z)dz = 0$$

***Example 4*** If $C$ is a circle $|z| = 1$ , prove that $\oint_C \frac{z^2}{z-3}dz = 0$

*Solution*

$$f(z) = \frac{z^2}{z-3} \quad , \quad z_0 = 3$$

$z_0$ تمثل نقطة او قطب ( شاذ ) ، اي ان الدالة تحليلية لجميع النقاط في المستوى العقدي عدا النقطة $z_0 = 3$ ، لانها تقع ( خارج الدائرة $|z| = 1$ ) وعليه فان قيمة التكامل تساوي صفر .

ملاحظــة : الدالة العقدية هي $\frac{z^2}{z-3}$ كما موضح في السؤال .

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**Example 5** If $C$ is $|z| = 2$ , evaluate $\int \frac{1}{z-3} dz$

*Solution*

$\int \frac{1}{z-3} dz$ , not define at $z = 3$ not in $C : |z| = 2$ , hence

$\int \frac{1}{z-3} dz = 0$

**Example 6** Find the integral $\oint_C \frac{dz}{(z-1)^2(z^2+4)}$ where $C$ is $3 \le |z| \le 5$ ,

*Solution*

$f(z) = \frac{1}{(z-1)^2(z^2+4)}$

$$(z-1)^2 = 0 \quad \rightarrow \boldsymbol{z = 1}$$

$$z^2 + 4 = 0 \rightarrow z^2 = -4 \rightarrow \boldsymbol{z = \mp 2i}$$

اذن الدالة تحليلية في جميع نقاط المستوى العقدي عدا النقاط

$$\boldsymbol{z = 1}, \quad \boldsymbol{z = +2i}, \quad \boldsymbol{z = -2i}$$

وجميعها **خارج المنطقة** وعليه فان **قيمة التكامل تساوي صفر** .

## نظرية موريرا ( Morera's Theorem )

وهي عكس نظرية كوشي – ريمان وتنص على الاتي :

أذا كانت $f(z)$ دالة مستمرة في المنطقة $\mathcal{R}$ وان $\oint f(z)dz = 0$ حيث $C$ هو منحني بسيط مغلق عندئذ تكون $f(z)$ دالة تحليلية في المنطقة $\mathcal{R}$ .

## نظرية ليوفيل ( Louvelli's Theorem )

اذا كانت $f(z)$ دالة كلية ( Entire function ) ومحددة لكل قيم $z$ في المستوى العقدي عندئذ تكون $f(z)$ دالة ثابتة او بعبارة اخرى ، افرض ان $f(z)$ دالة تحليلية محددة اي ان $|f(z)| < M$ لثابت ما $M$ لجميع قيم $z$ في المستوى العقدي الشامل ، ينتج ذلك ان $f(z)$ يجب ان تكون ثابتة .

## صيغ كوشي التكاملية ( Cauchy's integral formulas )

يمكن تقسيمها الى قسمين هما :

(1) ذات القطب البسيط ( Simple pole )

(2) ذات القطب المتعدد ( Multipole pole )

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اذا كانت $f(z)$ تحليلية على المنحني البسيط $C$ وداخلها وان $z_0$ اي نقطة داخل $C$ والحركة عكس عقرب الساعة ( تكون موجبة ) فان :

$$(1) \oint_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z)$$

$$(2) \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z)$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad (n = 0,1,2, \dots)$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \qquad (n = 1,2, \dots)$$

***Example 7*** Evalute $\oint_C \frac{e^z}{(z-1)} dz, \quad c : |z| = 2$

*Solution*

$$z_0 = 1$$

$$\oint_C \frac{e^z}{(z-1)} dz = 2\pi i f(z_0)$$

$$f(z) = e^z \rightarrow f(z_0) = e^{z_0} = e^1$$

$$\oint_C \frac{e^z}{(z-1)} dz = 2\pi i e^1$$

***Example 8*** Evalute $\oint_C \frac{\cos z}{(z+\pi)} dz, \quad c : |z| = 3$

$$f(z) = \cos z, \quad z_0 = -\pi, f(z_0) = \cos(-\pi)$$

$$\oint_C \frac{\cos z}{(z+\pi)} dz = 2\pi i \cos(-\pi) = -2\pi i$$

***Example 9*** Evalute $\oint_C \frac{e^z}{z^2} dz, \quad c : |z| = 2$

$$f(z) = e^z, \quad z_0 = 0, f(z_0) = f(z) = e^{z_0} = e^0 = 1$$

$$\oint_C \frac{e^z}{z^2} dz = 2\pi i (e^0) = 2\pi i$$

***Example 10*** Evalute $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz, \quad if \ c \ is : (a) \ |z| = 3, (b) \ |z| = 1$

$$(a) \ |z| = 3$$

$$z_0 = 2, f(z) = e^z, \qquad f(z_0) = e^2$$

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$\therefore \oint_C \frac{e^z}{z-2} dz = 2\pi i(e^2)$

$$\therefore \frac{1}{2\pi i} \times 2\pi i(e^2) = e^2$$

$(a)\ |z| = 2$

وحسب نظرية كوشي كورسات $z_0 = 2$ ، فان

$$\therefore \oint_C \frac{e^z}{z-2} dz = 0$$

لان الدالة $e^z$ تحليلية في جميع المناطق للمستوى العقدي ماعدا $z_0 = 2$ والتي تقع خارج المنحني $|z| = 1$ .

***Example 11*** Evalute $\oint_C \frac{\sin^2 z}{z-\pi/6} dz , \quad if \ \ c \ \ is : \_|z| = 1$

*Solution*

$$z_0 = \pi/6 \quad , f(z) = \sin^2 z$$

$$\therefore f(z) = \sin^2(\pi/6) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\oint_C \frac{\sin^2 z}{\sin(-\pi/6)} dz = 2\pi i\left(\frac{1}{64}\right) = \frac{\pi}{32}\boldsymbol{i}$$

***Example 12*** Evalute $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz , \quad if \ \ c \ \ is : \_|z| = 3$

*Solution*

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\oint \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = \oint \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)} dz - \oint \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)} dz$$

$$f(z) = \sin\pi z^2 + \cos\pi z^2 , \quad z_0 = 2 , \ z_0 = 1$$

$$f(z_0)_2 = \sin\pi(2)^2 + \cos\pi(2)^2 = 1$$
$$f(z_0)_1 = \sin\pi(1)^2 + \cos\pi(1)^2 = -1$$

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Then,

$$\oint \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i$$

لان $z_0 = 1$ ، $z_0 = 2$ داخل المنحني $C$ .

***Example 13*** Evalute $\oint_C \frac{cosz}{(z-\pi)^5} dz$

*Solution*

Since

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$f^{(4)}(z_0) = -sinz \underset{①}{} \rightarrow -cosz \underset{②}{} \rightarrow sinz \underset{③}{} \rightarrow cosz \underset{④}{}$$

$$z_0 = \pi \quad , n = 4$$

$$\oint_C \frac{cosz}{(z-\pi)^5} dz = \frac{2\pi i}{4*3*2*1} cos\pi = -\frac{\pi}{12} i$$

***Example 13*** Evalute $\oint_C \frac{e^z}{(z+1)^4} dz$ where $c$ is a circle $|z| = 3$

*Solution*

$$f(z) = e^{2z} \quad , z_0 = -1 \quad , \quad n = 3$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$f^{(3)}(z) = 8e^{2z} \quad , f^{(3)}(z_0) = f^{(3)}(-1) = 8e^{-2}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3*2*1} \times 8e^{-2} = \frac{8\pi i}{3} e^{-2}$$

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***Example 14*** Evalute $\oint_C \frac{e^{iz}}{z^3} dz$ where $c$ is a circle $|z| = 2$

*Solution*

$$z_0 = 0 \quad , \quad n = 3 \quad , \quad f(z) = e^{iz}$$

$$f'(z) = ie^{iz} \quad , \qquad f''(z) = -e^{iz} \quad , \qquad f''(z_0) = f''(0) = 1$$

$$\oint_C \frac{e^{iz}}{z^3} dz = \frac{2\pi i}{2 * 1}(-1) = -\pi i$$

$$\oint_C \frac{e^{iz}}{z^3} dz = -\pi i$$

# Complex Integration with Residue

## المحاضرة 13

### ✓ التكامل العقدي باستخدام الرواسب ( البواقي) ( Residue )

نفرض ان $f(z)$ دالة تحليلية على الدائرة $C$ وداخلها وان $z_0$ نقطة شاذة معزولة للدالة $f(z)$ ، فان مفكوك متسلسلة لورنت للدالة $f(z)$ هو :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (1)$$

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots . + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad , n = 0, \mp 1, \mp 2, \ldots \qquad (2)$$

عندما كانت $n = -1$ فان المعادلة (2) ،

$$a_{-1} = \frac{1}{2\pi i} \oint f(z)$$

يسمى $(a_{-1})$ راسب الدالة $f(z)$ عند النقطة $z_0$ ويرمز له $Res(f,z)$ وعليه فان له $Res(f,z) = a_{-1}$ ، لذلك يكون

$$\oint f(z)$$

كذلك اذا كان للدالة $f(z)$ قطب من الرتبة $n$ عند النقطة $z_0$ فان :

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{f(z)(z - z_0)^n\}$$

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**Example 1** Evaluate the integral $\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz$ where $C$ is a circle $|z-3|=1$ in positive direction.

*Solution*

$z_0=\pi$ القطب من الرتبة الثالثة

$$Res(f,z_0)=\frac{1}{(n-1)!}lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\left\{(z-\pi)^3\frac{e^{iz}-sinz}{(z-\pi)^3}\right\}$$

$$Res(f,z_0)=\frac{1}{2}lim_{z\to\pi}\frac{d^2}{dz^2}\{e^{iz}-sinz\}=\frac{1}{2}$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=2\pi i\ Res(f,\pi)=2\pi i*\frac{1}{2}=\pi i$$

$$\oint_C \frac{e^{iz}-sinz}{(z-\pi)^3}dz=\pi i$$

**Example 2** Use the residue theorem to find $\oint_C \frac{dz}{(z-1)(z+1)}$ where $C$: $|z|=3$ .

*Solution*

$$z_0=1\ \ ,z_0=-1$$

تمثلان قطبين بسيطين للدالة $f(z)$ ويقعان داخل الدائرة $|z|=3$ .

$$f(z)=\frac{1}{(z-1)(z+1)}$$

1. $z_0=1$

$$Res(f,1)=lim_{z\to1}\left\{(z-1)\frac{1}{(z-1)(z+1)}\right\}=\frac{1}{2}$$

2. $z_0=-1$

$$Res(f,-1)=lim_{z\to-1}\left\{(z+1)\frac{1}{(z+1)(z+1)}\right\}=\frac{-1}{2}$$

$$\oint_C \frac{dz}{(z-1)(z+1)}=2\pi i[Res(f,1)+Res(f,-1)]$$

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$$\oint_C \frac{dz}{(z-1)(z+1)} = 2\pi i\left[\frac{1}{2} - -\frac{1}{2}\right] = 0$$

$$\oint_C \frac{dz}{(z-1)(z+1)} = 0$$

***Example 3*** Find the residue of function $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ .

*Solution*

$$z_0 = -1 \quad , \quad z_0 = \mp 2i \quad , \qquad n = 2$$

$$z_0 = -1 \quad , \quad z_0 = \mp 2i \quad , \qquad n = 2$$

**1. Residue at $z_0 = -1$**

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to -1} \frac{d}{dz}\left\{(z+1)^2 \times \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$$

$$es(f,-1) = lim_{z\to -1}\left\{\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\right\} = -\frac{14}{25}$$

**2. Residue at $z_0 = 2i$**

$$Res(f, z_0) = \frac{1}{(n-1)!} lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\{f(z)(z-z_0)^n\}$$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to 2i} \frac{d}{dz}\left\{(z-2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f,-1) = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

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## 3. Residue at $z_0 = -2i$

$$Res(f,-1) = \frac{1}{1!} lim_{z\to -2i} \frac{d}{dz}\left\{(z+2i) \times \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$Res(f,-1) = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

***Example 4*** Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ where C is a circle $|z| = 3$ .

*Solution*

$$f(z) = \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$z_0 = 0 \quad , \quad z_0 = -1 \mp i \ , \qquad n = 2$$

قطب من الرتبة الثانية

*1.* $$Res(f,-1) = \frac{1}{1!} lim_{z\to 0} \frac{d}{dz}\left\{z^2 \times \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= lim_{z\to 0}\left\{\frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{z^2(z^2+2z+2)}\right\}$$

$$Res(f,-1) = \frac{t-1}{2}$$

2. Residue at $z_0 = -1+i$

$$Res(f,-1+i) = lim_{z\to -1+i}\left\{\{z-(-1+i)\} \times \frac{e^{zt}}{z^2\{z-(-1+i)\}\{z-(-1-i)\}}\right\}$$

$$Res(f,-1+i) = \frac{e^{(-1+i)t}}{(-1+i)^2} \times \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

3. Residue at $z_0 = -1-i$

$$Res(f,-1+i) = lim_{z\to -1-i}\left\{\{z-(-1-i)\} \times \frac{e^{zt}}{z^2\{z-(-1-i)\}\{z-(-1+i)\}}\right\}$$

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$$Res(f,-1+i)=\frac{e^{(-1-i)t}}{(-2i)(-1-i)^2}=\frac{e^{(-1-i)t}}{4}$$

مجموع الرواسب ،

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i$$

$$=2\pi i\left[\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right]$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i\left[\frac{t-1}{2}+\frac{e^{-t}}{2}cost\right]$$

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$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz=\frac{1}{2}(t-1)+\frac{1}{2}e^{-t}cost$$